Ph.D. QUALIFIER EXAMINATION: ANALYSIS
Winter 2012

Instructions: Answer *exactly* 6 of the 10 questions given. If you answer more than 6 questions, your grade will be determined by the first 6 questions that you answered. To pass this exam, you need to get 35 out of 60. Each question is graded out of 10.

Some Notation.

1. $\mathbb{R}^k$ – Euclidean k-dimensional space
2. $\mathbb{C}$ – the complex numbers
3. $(X, \mathcal{M}, \mu)$ – a measure space where X is a set, $\mathcal{M}$ is a σ-algebra of subsets of X, and μ is a measure on $\mathcal{M}$
4. a.e.$[\mu]$ – almost every with respect to the measure μ
5. m – Lebesgue measure on $\mathbb{R}^k$
6. $\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{1/p}$ – the L^p-norm of a μ-measurable function $f : X \to \mathbb{C}$
7. $\|f\|_\infty$ – the essential supremum of f
8. p, q – conjugate exponents where $\dfrac{1}{p} + \dfrac{1}{q} = 1$
9. $L^p(\mu)$ – the space of μ-measurable functions $f : X \to \mathbb{C}$ with $\|f\|_p < \infty$
10. $L^p(\mathbb{R}^k)$ – the space of Lebesgue measurable functions $f : \mathbb{R}^k \to \mathbb{C}$ with $\|f\|_p < \infty$
11. $\|\Gamma\| = \sup\{\|\Gamma x\| : x \in X, \|x\| \le 1\}$ – operator norm of a linear transformation $\Gamma : X \to Y$ where X and Y are normed linear spaces
12. $|\lambda|$ – the total variation of a measure λ.
13. $\lambda \ll \mu$ – the measure λ is absolutely continuous with respect to the measure μ
14. $\lambda \perp \mu$ – the measures λ and μ are mutually singular
15. $\dfrac{d\lambda}{d\mu}$ – the Radon-Nikodym derivative of λ with respect to μ where $\lambda \ll \mu$
16. Lip α – the space of complex functions f on $[a, b]$ for which $\sup\limits_{x \ne y} \dfrac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty$; here $0 < \alpha \le 1$
17. $f * g$ – the convolution of f and g: $(f * g)(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x - y) g(y) \, dy$
18. $C_0(\mathbb{R}^k)$ – the continuous complex functions on $\mathbb{R}^k$ which vanish at infinity
19. $\hat{f}(t) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x) e^{-ixt} \, dx$ – the Fourier transform

Questions

1. State and prove Lebesgue's Monotone Convergence Theorem.

2. Construct a sequence of continuous functions f_n on $[0,1]$ such that $0 \le f_n \le 1$, such that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0,$$

but such that the sequence $\{f_n(x)\}$ converges for no $x \in [0,1]$.

3. Assume that φ is a continuous real function on (a,b) such that

$$\varphi\left(\frac{x+y}{2}\right) \le \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(y)$$

for all x and y in (a,b). Prove that φ is convex.

4. Suppose that H is a Hilbert space with inner product $(\cdot,\cdot)$. Prove that if L is a continuous linear functional on H, then there exists a unique $y \in H$ such that $Lx = (x,y)$.

5. State and prove the Banach-Steinhaus Theorem. [You may assume Baire's Theorem in your proof.]

6. Prove that the total variation $|\mu|$ of a complex measure μ on $\mathcal{M}$ is a positive measure on $\mathcal{M}$.

7. Suppose that f is an absolutely continuous, nondecreasing function on $[a,b]$. Prove that if E is Lebesgue measurable with $m(E) = 0$, then $f(E)$ is Lebsegue measurable with $m(f(E)) = 0$.

8. Let $\mathcal{B}_k$ denote the σ-algebra of all Borel sets in $\mathbb{R}^k$. Prove that $\mathcal{B}_{m+n} = \mathcal{B}_m \times \mathcal{B}_n$.

9. Suppose that A and B are Lebesgue measurable subsets of $\mathbb{R}$, each having finite positive Lebesgue measure. Prove that $\chi_A * \chi_B$ is continuous and not identically equal to 0.

10. Suppose that $\{f_n\}$ is a uniformly bounded sequence of holomorphic functions on the region Ω such that $\{f_n(z)\}$ converges for every $z \in \Omega$. Prove that the convergence is uniform on every compact subset of Ω.