Ph.D. QUALIFIER EXAMINATION: ANALYSIS
Fall 2011

Instructions: Answer *exactly* 6 of the 10 questions given. If you answer more than 6 questions, your grade will be determined by the first 6 questions that you answered.

Some Notation.

1. $\mathbb{R}^k$ – Euclidean k-dimensional space
2. $\mathbb{C}$ – the complex numbers
3. $(X, \mathcal{M}, \mu)$ – a measure space where X is a set, $\mathcal{M}$ is a σ-algebra of subsets of X, and μ is a measure on $\mathcal{M}$
4. a.e.$[\mu]$ – almost every with respect to the measure μ
5. m – Lebesgue measure on $\mathbb{R}^k$
6. $\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}$ – the L^p-norm of a μ-measurable function $f : X \to \mathbb{C}$
7. $\|f\|_\infty$ – the essential supremum of f
8. p, q – conjugate exponents where $\dfrac{1}{p} + \dfrac{1}{q} = 1$
9. $L^p(\mu)$ – the space of μ-measurable functions $f : X \to \mathbb{C}$ with $\|f\|_p < \infty$
10. $L^p(\mathbb{R}^k)$ – the space of Lebesgue measurable functions $f : \mathbb{R}^k \to \mathbb{C}$ with $\|f\|_p < \infty$
11. $\|\Gamma\| = \sup\{\|\Gamma x\| : x \in X, \|x\| \leq 1\}$ – operator norm of a linear transformation $\Gamma : X \to Y$ where X and Y are normed linear spaces
12. $|\lambda|$ – the total variation of a measure λ.
13. $\lambda \ll \mu$ – the measure λ is absolutely continuous with respect to the measure μ
14. $\lambda \perp \mu$ – the measures λ and μ are mutually singular
15. $\dfrac{d\lambda}{d\mu}$ – the Radon-Nikodym derivative of λ with respect to μ where $\lambda \ll \mu$
16. Lip α – the space of complex functions f on $[a, b]$ for which $\sup_{x \neq y} \dfrac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty$; here $0 < \alpha \leq 1$
17. $f * g$ – the convolution of f and g: $(f * g)(x) = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - y) g(y) \, dy$
18. $C_0(\mathbb{R}^k)$ – the continuous complex functions on $\mathbb{R}^k$ which vanish at infinity
19. $\hat{f}(t) = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixt} \, dx$ – the Fourier transform

Questions

1. State and prove Lebesgue's Dominated Convergence Theorem. [You may assume Fatou's Lemma in your proof.]

2. Let X be a locally compact Hausdorff space in which every open set is σ-compact. If λ is a positive Borel measure on X such that $\lambda(K) < \infty$ for every compact subset K of X, then λ is regular. [Hint: there is a regular positive Borel measure μ on X such that $\int_X f \, d\lambda = \int_X f \, d\mu$ for all continuous f on X with compact support; show that $\lambda = \mu$. You may assume Urysohn's Lemma and the Monotone Convergence Theorem in your proof.]

3. For a positive measure μ prove that if $r < p < s$, then $\|f\|_p \leq \max\{\|f\|_r, \|f\|_s\}$ for every complex measurable function f. [Hint: for $\phi(p) = \|f\|_p^p$, the function $\log \phi(p)$ is convex in the interior of $\{p : \phi(p) < \infty\}$.]

4. Prove that if $A \subset [0, 2\pi]$ is Lebesgue measurable, then

$$\lim_{n \to \infty} \int_A \cos nx \; dx = \lim_{n \to \infty} \int_A \sin nx \; dx = 0.$$

5. Let X be a normed linear space, and X^* its dual space equipped with the norm $\|f\| = \sup\{|f(x)| : \|x\| \leq 1\}$. Prove that X^* is a Banach space.

6. Let $L^\infty = L^\infty(m)$ where m is Lebesgue measure on $I = [0, 1]$. Prove that there is a bounded linear functional $\Lambda \neq 0$ on L^∞ that is 0 on $C(I)$ (the space of continuous functions defined on I).

7. Prove that if $f \in \text{Lip } 1$ on $[a, b]$, then f is absolutely continuous on $[a, b]$ and $f' \in L^\infty$.

8. For $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ with $1 < p < \infty$, prove that $f * g$ exists a.e., that $f * g \in L^p(\mathbb{R})$, and that $\|f * g\|_p \leq \|f\|_1 \|g\|_p$. [You may assume Fubini's Theorem in your proof.]

9. For a positive integer n, find (with proof) the Fourier transform of $\chi_{[-n,n]} * \chi_{[-1,1]}$, where $\chi_{[a,b]}$ is the characteristic function of $[a, b]$.

10. Suppose f is an entire function, and that in every power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n,$$

at least one coefficient is 0. Prove that f is a polynomial.