May 2016- Ph.D. Preliminary Examination
Ordinary Differential Equations

Instructions: Give solutions to 7 of the following 8 problems.

(1) Consider the system

$$\dot{x}_1 = -2x_2 + x_2x_3,$$
$$\dot{x}_2 = x_1 - x_1x_3,$$
$$\dot{x}_3 = x_1x_2.$$

Prove or disprove: the equilibrium at the origin is asymptotically stable. [Hint. Is $V(x_1, x_2, x_3) = c_1x_1^2 + c_2x_2^2 + c_3x_3^2$ a Lyapunov function for some choice of c_1, c_2, c_3?]

(2) The system

$$\dot{x} = -y + x(1 - x^2 - y^2),$$
$$\dot{y} = x + y(1 - x^2 - y^2),$$

has a periodic solution $\gamma(t) = (\cos t, \sin t)$. By switching to polar coordinates, find a Poincaré section, and use the corresponding Poincaré map to determine the stability of the periodic solution $\gamma(t)$.

(3) For $x \in \mathbb{R}^n$, suppose $\dot{x} = Ax + g(x)$ where A is a real $n \times n$ matrix all of whose eigenvalues are negative, and there exist $a > 0$ and $k > 0$ such that $\|g(x)\| \leq k\|x\|^2$ for all $\|x\| < a$. Prove there exists $C > 0$, $b > 0$, and $\alpha > 0$ such that

$$\|x(t)\| \leq C\|x(0)\|e^{-\alpha t} \text{ for all } t \geq 0,$$

whenever $\|x(0)\| \leq b$.

(4) The Rectification Lemma for $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ where f is C^1, states that if $f(p) \neq 0$ for $p \in \mathbb{R}^n$, then there exists open sets U and V in $\mathbb{R}^n$ with $p \in U$ and a diffeomorphism $g : U \to V$ such that with $g(x) = y$, the equation $\dot{x} = f(x)$ becomes $\dot{y} = Dg(g^{-1}(y))f(g^{-1}(y))$ where $Dg(g^{-1}(y))f(g^{-1}(y)) = (1, 0, 0, \ldots, 0)$. Prove the Rectification Lemma.

(5) Let $f : X \to X$ be continuous where X is a complete metric space. Show that if q is an ω-limit point of $p \in X$ where q is periodic, then the ω-limit set of p is the orbit of p.

(6) Let $E_3 : S^1 \to S^1$ be the map $E_3(x) = 3x \mod 1$. Prove that this map is topologically mixing.

(7) Suppose $x \in [0, 1]$ does not have a periodic or eventually periodic orbit under the *tent map*

$$T(x) = \begin{cases} 2x & 0 \leq x < 1/2 \\ 2(1-x) & 1/2 \leq 1. \end{cases}$$

Does the orbit of x have to be dense in $[0, 1]$? If so prove it. If not, give an example. [Hint: Find a conjugacy with $(\sigma, \sum_2^+)$, the shift map on two symbols.]

(8) Suppose $f : X \to X$ and $g : Y \to Y$ are continuous functions where X and Y are metric spaces. If there is a semi-conjugacy $\pi : X \to Y$ from f to g, prove that the topological entropy $h(f) \geq h(g)$.