WINTER 2016 - PH.D. PRELIMINARY EXAMINATION ORDINARY DIFFERENTIAL EQUATIONS AND DYNAMICAL SYSTEMS

Instructions: Give solutions to exactly 6 of the following 8 problems. If you give more than 6 solutions, your grade will be determined by the first six that appear.

(1) Sketch the phase portraits of the following equations and determine the stability of equilibrium points.

$$\begin{aligned} x' &= y \\ y' &= -x + x^3 - \epsilon y \end{aligned}$$

when $\epsilon < 0$, $\epsilon = 0$, and $\epsilon > 0$.

(2) Let A be a 2×2 matrix with one positive eigenvalue and one negative eigenvalue and $g(t)$ be a bounded continuous function from $(-\infty, \infty)$ to $\mathbb{R}^2$. Prove that

$$x' = Ax + g(t)$$

has an unique bounded solution over $(-\infty, \infty)$.

(3) Let $A(t)$ be a continuous family of $n \times n$ matrices and let $P(t)$ be the matrix solution to the initial value problem $\frac{d}{dt}P = A(t)P, P(0) = P_0$. Show that

$$\det P(t) = (\det P_0) \exp\left(\int_0^t \mathrm{Tr} A(s) ds\right).$$

(4) Use the Poincare-Bendixson Theorem to prove the Brouwer fixed point theorem for a C^1 mapping.

(5) Answer the following.
  (a) Define shadowing and state the shadowing theorem.
  (b) Use shadowing to prove that periodic points are dense for transitive Anosov diffeomorphisms.

(6) Answer the following:
  (a) Define *topological transitivity.*
  (b) Define *topologically mixing.*
  (c) Prove that if $f \in \mathrm{Diff}(M)$ and M contains a hyperbolic fixed point p for f such that $\overline{W^s(p)} = \overline{W^u(p)} = M$, then f is topologically transitive.

(7) Define a *subshift of finite type.* Let A be an eventually positive transition matrix and Σ_A be the subshift of finite type associated with A.
  (a) Prove that the periodic points of σ are dense in Σ_A.
  (b) Prove that σ has a point with a dense forward orbit.

(8) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as $f_\mu(x) = \mu x(1-x)$. Is it possible for $f_\mu(x)$ to undergo a saddle-node bifurcation at $x = 2/3$ for $\mu = 3$. Prove your answer.