AUGUST 2015- PH.D. PRELIMINARY EXAMINATION
ORDINARY DIFFERENTIAL EQUATIONS

Instructions: Give solutions to 7 of the following 8 problems.

(1) Consider the system

$$x' = x^2 + y$$
$$y' = x - y + \epsilon$$

where $\epsilon \in \mathbb{R}$ is a parameter.
  (a) Find all equilibrium points and compute the linearized equation at each.
  (b) Describe the behavior of the linearized system at each equilibrium point and determine the stability of equilibrium points.

(2) Find a bifurcation point for the following equations and a periodic solution for $\epsilon > 0$.

$$x' = y - x(x^2 + y^2 - \epsilon)$$
$$y' = -x - y(x^2 + y^2 - \epsilon)$$

(3) Let $A(t)$ be a continuous function from $\mathbb{R}$ to the space of $n \times n$ square, real-valued matrices. Show that for every solution of the (non-autonomous) linear system, $\dot{x} = A(t)x$ we have

$$||x(t)|| \leq ||x(0)||e^{\int_0^t ||A(s)||ds},$$

where $||A(s)||$ is the operator norm and $||x(t)||$ is the usual Euclidean norm.

(4) Consider the differential equation $\dot{x} = f(x)$ on the plane, where f is a smooth function. Suppose that there are two nonconstant periodic solutions for which one surrounds another and the region bounded by these two periodic solutions contains neither a periodic solution nor an equilibrium. Prove that these two periodic solutions cannot both be orbitally stable.

(5) A solution $x(t)$ of a system of differential equations is called recurrent if $x(t_n) \to x(0)$ for some sequence $t_n \to \infty$. Prove that a gradient dynamical system has no nonconstant recurrent solutions.

(6) Let U be an open set of $\mathbb{R}^n$ containing x_0. Suppose that $f : U \to \mathbb{R}^n$ is C^1 and $f(x_0) = 0$. Suppose further that there is a C^1 function $V : U \to \mathbb{R}$ satisfying $V(x_0) = 0$ and $V(x) > 0$ if $x \neq x_0$. Prove
  (a) if $\dot{V}(x) = \text{grad}V(x) \cdot f(x) \leq 0$ for all $x \in U$, then x_0 is stable.
  (b) if $\dot{V}(x) < 0$ for all $x \in U - \{x_0\}$, then x_0 is asymptotically stable.

(7) Let $f_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, \cdots$, satisfy for some constant $K > 0$,

$$|f_k(x) - f_k(y)| \leq K|x - y|, \qquad \text{for } x, y \in \mathbb{R}^n, \quad k = 1, 2, \cdots$$

and

$$\lim_{k\to\infty} f_k(x) = f(x), \qquad \text{uniformly.}$$

Let $x_k(t)$ be the solution of $\dot{x} = f_k(x),\ x(0) = x_0$ and $x(t)$ be the solution of $\dot{x} = f(x),\ x(0) = x_0$. Prove

$$\lim_{k\to\infty} x_k(t) = x(t), \qquad \text{uniformly.}$$

(8) Consider the following differential equation

$$x' = Ax + f(x)$$

where $x \in \mathbb{R}^2$, A is a 2×2 matrix, and $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a C^1 function with $f(0) = 0$ and $f'(0) = 0$. Assume that A has a positive eigenvalue. Prove that $x = 0$ is unstable.