FALL 2011 - PH.D. PRELIMINARY EXAMINATION
ORDINARY DIFFERENTIAL EQUAITONS

Instructions: Give solutions to exactly 6 of the following 8 problems. If you give more than 6 solutions, your grade will be determined by the first 6 that appear.

(1) State and prove either the Cauchy-Peano Existence Theorem or the Picard-Lindelöf Existence and Uniqueness Theorem for initial value problems. You may assume the Arzelá-Ascoli Theorem and the Contraction Mapping Principle.

(2) Prove that the origin is a uniformly asymptotically stable solution for the equation

$$\tfrac{dx}{dt} = y$$

$$\tfrac{dy}{dt} = -x - y^3(1 - 5\sin t).$$

(3) Prove that if e^{tA} and e^{tB} are contractions on $\mathbb{R}^n$, and $AB = BA$, then $e^{t(A+B)}$ is a contraction.

(4) Let $\varphi_t(x, y)$ be a smooth flow on $\mathbb{R}^2$. Let $p \in \mathbb{R}^2$ such that $p \in \omega(p)$ under the flow. Prove that p is fixed or periodic.

(5) Let A be an irreducible matrix whose entries are 0s and 1s (so $a_{ij} \in \{0, 1\}$ for all i and j). Let Σ_A be the subshift of finite type associated with A.
(a) Prove that Σ_A is transitive.
(b) Prove that the topological entropy of Σ_A is

$$\lim_{n\to\infty} \frac{\log |w_n(\Sigma_A)|}{n}$$

where $w_n(\Sigma_A)$ is the set of words of length n in Σ_A.

(6) Find a bifurcation point for the following equation and a periodic solution for $\epsilon > 0$.

$$x' = y - x(x^2 + y^2 - \epsilon)$$
$$y' = -x - y(x^2 + y^2 - \epsilon)$$

(7) Answer the following.
   (a) Define shadowing and state the shadowing theorem.
   (b) Use shadowing to prove that periodic points are dense for transitive Anosov diffeomorphisms.

(8) (a) State the Hartman-Grobman Theorem.
   (b) Consider the following pairs of linear functions of $\mathbb{R}^2$ to $\mathbb{R}^2$ given by the following matrices. For each pair tell whether the corresponding linear functions are topologically conjugate. Give reasons for your answer.
      (i) $\begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$ and $\begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$
      (ii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
      (iii) $\begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}$ and $\begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 1/3 \end{bmatrix}$