WINTER 2010 - PH.D. PRELIMINARY EXAMINATION
ORDINARY DIFFERENTIAL EQUATIONS

Instructions: Give solutions to exactly 6 of the following 8 problems. If you give more than 6 solutions, your grade will be determined by the first six that appear.

(1) Answer the following.
  (a) Define a hyperbolic invariant set.
  (b) Define an ϵ-chain.
  (c) State the Shadowing Lemma.
  (d) Assume that $\mathcal{R}(f)$ (the chain recurrent set) is hyperbolic. Prove that the periodic points are dense in $\mathcal{R}(f)$.

(2) Define *rotation number* for a circle homeomorphism and show that the rotation number for an orientation preserving circle homeomorphism is rational if and only if there exists a periodic point.

(3) Let f be a diffeomorphism and p a periodic point for f. Let H_p be the equivalence class of periodic points heteroclinically related to p, i.e. $q \sim p$ if $W^s(p) \pitchfork W^u(q) \neq \emptyset$ and $W^s(q) \pitchfork W^u(p) \neq \emptyset$. Let Λ_p be the closure of H_p. Prove that Λ_p is transitive.

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 5x(1-x)$. Describe, up to topological conjugacy, the dynamics of the set of points x such that $f^n(x) \in [0,1]$ for all $n \geq 0$. Prove your answer.

(5) Let x be a recurrent point of a C^1 2-d system: $x' = f(x)$. Prove that either x is an equilibrium point or x lies on a closed orbit.

(6) Consider a differential equation $x' = Ax + f(x)$, where $x \in \mathbb{R}^n$, A is a $n \times n$ matrix with a positive eigenvalue, f is a C^1 function from $\mathbb{R}^n$ to $\mathbb{R}^n$ with $f(0) = 0$ and $f'(0) = 0$. Prove that $x = 0$ is unstable.

(7) Let f be a C^1 vector field on a neighborhood of the annulus

$$A = \{x \in \mathbb{R} \mid 1 \leq |x| \leq 2\}.$$

Suppose that f has no zeros and f is transverse to the boundary, pointing inward. Prove there is a closed orbit for $x' = f(x)$.

(8) Consider $x' = f(x)$ and its perturbed equation $x' = f(x) + \epsilon h(t,x)$ where $x \in \mathbb{R}^n$, f and h are C^2 functions, and $h(t,x)$ is $T-$periodic in t, $0 < \epsilon$ is a parameter. Prove that if $x' = f(x)$ has a hyperbolic equilibrium point p^*, then the perturbed equation $x' = f(x)+\epsilon h(t,x)$ has a unique periodic solution $p(t,\epsilon)$ such that

$$p(t,\epsilon) - p^* = O(\epsilon).$$