1. Let $f : X \to Y$ be a surjective continuous function of topological spaces. Suppose that Y is connected and $f^{-1}(y)$ is connected for every $y \in Y$.
   (a) Give an example where X is not connected.
   (b) Show that if f is a quotient map, then X is connected.

2. Let $f : \mathbb{RP}^{2n} \to Y$ be a covering map where Y is a CW-complex.
   (a) Show that the Euler characteristic of Y divides the Euler characteristic of $\mathbb{RP}^{2n}$.
   (b) Conclude that f is a homeomorphism.

3. Let $X = \mathbb{S}^2 \cup \{(0, 0, x) \in \mathbb{R}^3 \mid -1 \leq x \leq 1\}$ where $\mathbb{S}^2$ is the unit 2-sphere in $\mathbb{R}^3$. Compute the homology groups of X.

4. Let M be a closed orientable manifold M of dimension $2k$. Prove that if $H_{k-1}(M, \mathbb{Z})$ is torsion free, then $H_k(M, \mathbb{Z})$ is torsion free.

5. Let M be a compact manifold without boundary, and suppose that $\omega \in \Omega^1(M)$ is a closed nonvanishing 1-form on M. Prove that $H^1_{dR}(M) \neq 0$.

6. Let E be a smooth rank n vector bundle over the manifold M, with projection $\pi : E \to M$. Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ be a collection of local trivializations $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ which cover M. Let $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$ be the transition function from (U_β, φ_β) to $(U_\alpha, \varphi_\alpha)$.

   Prove that E is trivial if and only if for every $\alpha \in A$ there is a map
   $$\lambda_\alpha : U_\alpha \to GL(n, \mathbb{R})$$
   such that $\tau_{\alpha\beta}(x) = \lambda_\alpha(x)^{-1} \cdot \lambda_\beta(x)$ for all $\alpha, \beta \in A$ and $x \in U_\alpha \cap U_\beta$.

7. Suppose that M is a smooth k manifold. Prove that any smooth map $F : M \to S^n$ with $n > k$ is homotopic to a constant map.

8. Let $S^1 = \{e^{2\pi it} \in \mathbb{C} \mid t \in [0, 1)\}$, and consider the map
   $$\begin{aligned} F : S^1 &\to S^1 \times S^1 \\ e^{2\pi it} &\mapsto \left(e^{3\cdot(2\pi it)}, e^{2\cdot(2\pi it)}\right). \end{aligned}$$
   Prove that F is an embedding. Describe (in words) its image.