1. Prove that every perfect subset of a complete metric space is uncountable.

2. Prove that $\tilde{H}_n(X) = \tilde{H}_{n+1}(SX)$ where SX is the suspension of X, i.e. $SX = X \times [0,1]/\sim$ where $(x,t) \sim (y,t)$ for all $x, y \in X$ and $t \in \{0,1\}$.

3. Let X be a path-connected, locally path-connected topological space with finite fundamental group and $\mathbb{S}^k$ the unit sphere in $\mathbb{R}^{k+1}$.

   (a) Prove that any continuous map $f : X \to \mathbb{S}^1 \times \mathbb{S}^1$ is homotopic to a constant map.

   (b) Show that the quotient map $q : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^2$ obtained by collapsing

   $$\mathbb{S}^1 \times \{x_0\} \vee \{x_0\} \times \mathbb{S}^1$$

   to a point for $x_0 \in \mathbb{S}^1$ is not nullhomotopic.

4. Let X_i be a CW complex for $i \in \{1, \cdots, n\}$ and $\mathbb{S}^k$ the unit sphere in $\mathbb{R}^{k+1}$.

   (a) Prove that $\tilde{H}_*(\bigvee_{i=1}^{n} X_i) = \bigoplus_{i=1}^{n} \tilde{H}_*(X_i)$.

   (b) Compute $H_*(\mathbb{S}^1 \vee \mathbb{S}^2)$.

5. Let M_n be the space of $n \times n$ real matrices, with topology obtained through the natural identification of M_n with $\mathbb{R}^{n^2}$. Let $S_n \subset M_n$ be the subspace of symmetric matrices, and let $F : M_n \to S_n$ be the smooth map given by $F(A) = AA^T$.

   (a) Show that map induced by F on the tangent space

   $$d_A F : T_A M_n \longrightarrow T_{F(A)} S_n$$

   is given by $d_A F(B) = AB^T + BA^T$. Note that here we are using the identification

   $$T_A M_n \cong \mathbb{R}^{n^2} \cong M_n(\mathbb{R}).$$

   (b) Show that the $n \times n$ identity matrix $\mathrm{Id} \in S_n$ is a regular value of the map F.

   (c) Show that $O(n) = F^{-1}(\mathrm{Id})$ is a smooth submanifold of M_n, and determine its dimension.

   (d) Determine the tangent space of $O(n)$ at $\mathrm{Id} \in O(n)$. (Describe your answer as a familiar subspace of M_n.)

6. Let $M \subset \mathbb{R}^n$ be a smooth submanifold of dimension $m < n-2$. Show that the complement $\mathbb{R}^n \backslash M$ is both connected and simply-connected.

7. Define the deRham cohomology groups $H^i_{dR}(M)$ of a smooth manifold M. Use this definition to directly compute $H^i_{dR}(S^1)$ for $i = 0, 1, 2, \ldots$, where $S^1 = \mathbb{R}/\mathbb{Z}$.

8. Let M be a smooth compact 3-manifold with boundary $\partial M \neq \emptyset$. If θ_1 and θ_2 are two 1-forms on ∂M so that $\theta_1 \wedge \theta_2$ is a volume form on ∂M, show that θ_1 and θ_2 cannot both extend to closed forms on M. Can you find such M, θ_1, and θ_2 so that θ_1 extends to a closed 1-form on M?