Topology Qualifying Exam - Fall 2016

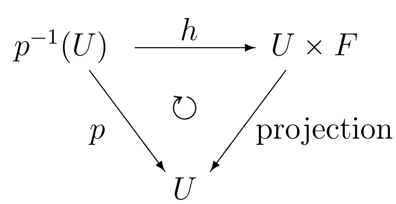
Instructions: Do all seven problems. Do not make assumptions or use known theorems which trivialize a problem.

(1) Let X, Y, and Z be a path connected and locally path connected, with $q : X \to Y$ and $r : Y \to Z$ continuous.

   (a) Show that if r and $r \circ q$ are covering maps, then so is q.
   (b) Show that if q and $r \circ q$ are covering maps, then so is r.

(2) Let X be a linearly ordered set with the order topology. Show that if X has the supremum property (every nonempty set which is bounded above has a supremum), then closed intervals of X are compact.

(3) Let E be a fiber bundle over B with fiber F. That is, there is a continuous surjection $p : E \to B$ with the property that for any $e \in E$ there exists a neighborhood U of $p(e)$ in B and a homeomorphism $h : p^{-1}(U) \to U \times F$ such that the following diagram commutes:



Prove that if F is path connected and $p(e_0) = b_0$, then $P_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is onto.

(4) Consider the submanifold M of $\mathbb{R}^3$ defined by $x^2 + y^2 - z^2 = 1$.

   (a) Show that the vector field $X = \dfrac{xz}{1+z^2}\dfrac{\partial}{\partial x} + \dfrac{yz}{1+z^2}\dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}$ is tangent to M.

   That is, there is a vector field Y on M such that $i_*(Y(m)) = X(m)$, $\forall\, m \in M$.

   (b) Show that the two form $w = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$ restricts to a nonvanishing form on M. (Hint: cylindrical coordinates.)

   (c) Show that the flow of Y on M preserves $i^*(w)$.

(5) Let A be the union of two once linked embedded circles in S^3. Let B be the union of two unlinked circles in S^3. Show that the cohomology groups of $S^3 - A$ and $S^3 - B$ are isomorphic, but the cohomology rings are not.

(6) Let M and N be closed n-manifolds and P be the connected sum of M and N. Show $\chi(P) = \chi(M) + \chi(N) - 2$ if n is even and $\chi(P) = \chi(M) + \chi(N)$ if n is odd ($\chi =$ Euler characteristic).

(7) (a) Suppose ω is a smooth exact k-form. Show that $\omega \wedge \omega$ is an exact $2k$-form.

   (b) Suppose ω is a smooth 2-form on S^4. Show that $\omega \wedge \omega$ vanishes somewhere.