Topology qualifying exam - Spring 2016

Written by Mark Hughes and Greg Conner

Instructions: Do all eight problems. Do not make assumptions or use known theorems which trivialize a problem.

(1) Let (X, τ) be a compact Hausdorff topological space.
   (a) Prove that any topology on X which is strictly coarser than τ cannot be Hausdorff.
   (b) Prove that any topology on X which is strictly finer than τ cannot be compact.

(2) Let A and B be open sets of the topological space X which cover X and whose intersection is simply connected and contains the point x_0. Show that if $\pi_1(X, x_0) \cong H_1(X;\mathbb{Z})$ then $\pi_1(A, x_0) \cong H_1(A;\mathbb{Z})$ and $\pi_1(B, x_0) \cong H_1(B;\mathbb{Z})$.

(3) Let Y be the wedge of two copies of S^1, the figure eight. Find two non-homeomorphic finite covering spaces of Y the of same degree. Find the corresponding subgroups of $\pi_1(Y)$, and show that they are distinct but isomorphic.

(4) Find all homology and cohomology groups of $T^2 \vee S^2$ (i.e. the wedge sum of the 2-torus T^2 and the 2-sphere S^2).

(5) Let W be finite simplicial complex which is a compact orientable manifold of dimension n whose boundary, ∂W, is a finite subcomplex. The *double* of W, denoted by DW, is the quotient space $(W \times \{0,1\})/\sim$, where $(x, 0) \sim (x, 1)$ for all $x \in \partial W$.

   Recall that if Y is a topological space then $\chi(Y)$ denotes the *Euler characteristic* of Y.

   Compute (with proof) a formula for $\chi(DW)$ in terms of $\chi(W)$ and $\chi(\partial W)$ and use it to prove that

$$2\chi(W) = \begin{cases} \chi(DW), \text{ if } n \text{ is even.} \\ \chi(\partial W), \text{ if } n \text{ is odd.} \end{cases}$$

   Hint: For a given W, either DW has odd dimension or ∂W has odd dimension. Don't assume any theorems about the Euler characteristic of a manifold; prove them if you need them.

(6) Let M be a closed compact smooth manifold, and suppose that θ is a smooth 1-form on M, which is nonzero everywhere on M. Prove that $H^1_{dR}(M) \neq 0$.

(7) Prove that the n-sphere S^n does not embed in $\mathbb{R}^n$ for any $n \geq 0$.

(8) Prove that every smooth map from S^2 to T^2 has degree zero.