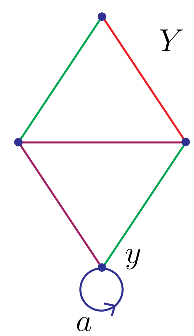
JANUARY 2016 TOPOLOGY QUALIFYING EXAM

(1) Construct a 4-fold covering space (X, x) of the following graph (Y, y) with the property that $a^2 \in Im(\pi_1(X, x) \hookrightarrow \pi_1(Y, y))$, but $a \notin Im(\pi_1(X, x) \hookrightarrow \pi_1(Y, y))$.



(2) Let X be a compact metric space and $f : X \to X$ an isometric embedding. Show f is onto.

(3) Let X be a Hausdorff space and $\{f_\alpha\}_{\alpha \in J}$ a family of continuous functions $f_\alpha : X \to \mathbb{R}$ satisfying the requirement that $\forall\, x \in X$ and $\forall\, U$ neighborhood of x, there exists $\alpha \in J$ with $f_\alpha(x) > 0$ and $f(y) = 0\ \forall\, y \notin U$. Prove that $F : X \to \mathbb{R}^J$ defined by $F(x) = (f_\alpha(x))_{\alpha \in J}$ is an embedding.

(4) Show that $P^2 \# P^2 \# P^2 \cong T \# P^2$ where T is the torus and P^2 the projective plane.

(5) Let M be an n-dimensional smooth closed compact manifold, and let TM be the total space of its tangent bundle. If $H^k_{dR}(TM)$ denotes the kth de Rham cohomology space of TM, prove that

$$H^k_{dR}(TM) \cong 0$$

for all $k > n$.

(6) Prove that for any smooth compact manifold M of dimension $n \geq 1$, there exists a smooth map $f : M \longrightarrow S^n$ of degree 1.

(7) Prove that there does not exist any immersion $f : S^2 \to \mathbb{R}^2$.

(8) Consider the vector fields

$$V = -y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y} + \frac{\partial}{\partial z} \qquad \text{and} \qquad W = a\,\frac{\partial}{\partial x} + b\,\frac{\partial}{\partial y} + c\,\frac{\partial}{\partial z}$$

on $\mathbb{R}^3$, where a, b, and c are real constants. Let φ_s and ψ_t denote the global flows of V and W respectively.

(a) What geometric transformations do φ_s and ψ_t correspond to?

(b) For what values of a, b, and c do φ_s and ψ_t commute, i.e. $\varphi_s \circ \psi_t = \psi_t \circ \varphi_s$ for all s and t?

(9) Let A be the subset of $\mathbb{R}^3$ consisting of the spheres of radius 1 and 2 about 0 together with a circle of radius 2 centered at $(0, 0, 2)$ passing through the origin. Find the homology groups of A with $\mathbb{Z}$ coefficients in simplest form.

(10) Let M be a compact orientable 7-dimensional manifold without boundary. Suppose $a \in H_2(M)$ $a \neq 0$, but $8a = 0$. Show that there is $b \in H_y(M)$ $b \neq 0$, but $8b = 0$.

(11) Let X be a finite simplicial complex. Find $H^*(X, \mathbb{Z}_8)$ and $H_*(X, \mathbb{Z}_8)$ given that

$$H_P(X, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } p = 0, 4 \\ \mathbb{Z}_2 & \text{if } p = 1, 5 \\ \mathbb{Z}_6 \oplus \mathbb{Z}_2 & \text{if } p = 2 \\ \mathbb{Z}_{18} \oplus \mathbb{Z}_{24} \oplus \mathbb{Z} & \text{if } p = 3 \\ 0 & \text{if } p > 5 \end{cases}$$

(12) Let A and B be disjoint embeddings of S^1 into $\mathbb{R}^3$. Find $H_*(\mathbb{R}^3 - (A \cup B))$.