Topology PhD Qualifying Exam - Winter 2014

If you think a problem is stated badly, state precisely what you think is intended, but do not interpret a question so that it becomes trivial. If it is false, provide a counterexample.

1. Recall that for M a smooth manifold, a Riemannian metric g on M is a family of inner products $g_m : M_m \times M_m \to \mathbb{R}$ on the tangent spaces of M such that for any X, Y smooth vector fields on M, the function $p \to g_p(X_p, Y_p)$ is smooth. Prove that every smooth manifold has a Riemannian metric.

2. Show that a compact Hausdorff space is normal.

3. Let M be a handle body of genus g (so M is the compact subset of $\mathbb{R}^3$ bounded by a standard embedding of a surface of genus g). Let X be the 3-manifold obtained by gluing 2 copies of M along the boundary via the identity map. Compute $H_*(X; \mathbb{Z})$ and $H_*(M, \partial M; \mathbb{Z})$.

4. Consider the map $f : \mathbb{S}^2 \to \mathbb{R}^6$ defined by

$$f(x, y, z) = (x^2, y^2, z^2, xy, xz, yz)$$

   - Show that f is an immersion.
   - Show that f induces an embedding on $\mathbb{RP}^2$.

5. Let X and Y be path connected locally contractable spaces with $H^1(X; \mathbb{Q}) \neq 0 \neq H^1(Y; \mathbb{Q})$. Show that $X \vee Y$ is not a retract of $X \times Y$.

6. Let D^2 be the closed unit disk in the complex plane. Let A, B, and C be the open disks of radius $\frac{1}{8}$ about the points $\frac{-1}{2}$, 0 and $\frac{1}{2}$ respectively. Let Z be the space obtained from $D^2 - (A \cup B \cup C)$ by identifying all four boundary circles so that their counterclockwise orientations are preserved. Compute the fundamental group of Z.

7. Let M be a smooth compact orientable n-manifold (without boundary). Show that for any $(n-1)$-form ω on M, there is point $p \in M$ such that $d\omega(p) = 0$.

8. Let X be a compact metric space. For $\epsilon > 0$, $A \subset X$ is an ϵ-net if for any $a, b \in A$, $d(a, b) \geq \epsilon$.
   - Show that any ϵ-net in X is finite.
   - Show that X has a countable dense subset.