FALL 2013 TOPOLOGY QUALIFYING EXAM

PROFS. CONNER AND PURCELL

Rules: Show all your work. If you think you've found a mistake in a problem or you've found a way to construe a problem so it is trivial or extremely easy, reconstrue the problem in such a way that it is meaningful, writing down your specific interpretation, before working it. Do 7 out of 8 problems.

(1) A map is *monotone* if the preimage of each point in the codomain is connected. Show that if X is a simply connected CW-complex then no continuous monotone map from X to S^1 is surjective by applying the *lifting criterion* to the universal cover of S^1.

(2) Compute the fundamental groups of: (a) the wedge of two projective planes and (b) the mapping torus over the antipodal map of S^1. Are these fundamental groups isomorphic?

(3) The i-th Betti number of a space X is the $\mathbb{Z}$-rank of $H_i(X, \mathbb{Z})$ and is denoted $\beta_i(X)$. Let M be a finite simplicial complex which is an orientable 4-manifold whose fundamental group is finite of order n and let $\tilde{M}$ be the universal covering space of M. Use two formulations of the *Euler characteristic* to show that $\beta_2(\tilde{M}) = n \cdot \beta_2(M) + 2n - 2$.

(4) Let Z be a genus 2 handlebody (that is, a "solid" two-holed surface or, equivalently, a regular neighborhood of a *figure 8* in $\mathbb{R}^3$). For all k compute $H_k(Z), H_k(\partial Z), H_k(Z, \partial Z)$ and $H^k(Z), H^k(\partial Z), H^k(Z, \partial Z)$.

(5) Let F be a *figure 8*, a wedge of 2 circles, embedded in S^3. Note that F is possibly *knotted*, that is, not isotopic to the standard figure 8. Compute $H_1(S^3 - F)$.

(6) Suppose M is a smooth n–manifold, and the smooth map $f : M \to \mathbb{R}^{2n+1}$ is injective. Let B be a closed subset of M, U an open subset containing B, and ϕ_B a smooth bump function for B supported in U. For any $b \in \mathbb{R}^{2n+1}$, define $g_b(x) : M \to \mathbb{R}^{2n+1}$ by

$$g_b(x) = f(x) + b\,\phi_B(x).$$

Show that there is some $b \neq 0$ in $\mathbb{R}^{2n+1}$ so that g_b is injective.

(7) Let ω be an exact 2-form on $M = S^3 \times S^5$ and let $g : S^1 \times S^1 \to M$ be a smooth map. Prove that $\int_{S^1 \times S^1} g^*\omega = 0$.

(8) Recall that the set of 2×2 real matrices, $\mathbb{R}^{2\times 2}$, is a smooth manifold diffeomorphic to $\mathbb{R}^4$. We see that $\mathbb{R}^{2\times 2}$ can be naturally partitioned into 3 subsets: the matrices of rank 2, the matrices of rank 1, and the matrices of rank 0. Show that each of the partition elements is a smooth submanifold of $\mathbb{R}^{2\times 2}$ and determine its dimension.