TOPOLOGY QUALIFYING EXAM
SPRING 2013

Instructions: Show all your work. If you think you've found a mistake in a problem or you've found a way to construe a problem so it is trivial or extremely easy, reconstrue the problem in such a way that it is meaningful, writing down your specific interpretation, before working it. Do 7 out of 8 problems.

(1) Let X be a space obtained by gluing all 3 sides of a solid (2-dimensional) triangle together to one edge via homeomorphisms sending vertices to vertices. Show that there are two such possibilities for X up to homeomorphism. Compute the corresponding fundamental groups. Compute all of the covering spaces of such spaces.

(2) Suppose X is a space which is covered by a simply connected, locally path connected, compact space. Suppose G is a torsion-free group and Y is a $K(G,1)$ space (i.e., Y has a universal cover which is contractible and $\pi_1(Y) = G$.) Show that any continuous map from X to Y is contractible (homotopic to a constant map). Partial credit will be given for completely working out the case where $X = \mathbb{R}P^2$ and $Y = S^1 \times S^1 \times S^1$.

(3) For all k compute $H_k(Z, \partial Z)$ where $Z = D^2 \times S^1$.

(4) Show that if S^n admits a nowhere vanishing vector field, then n is odd.

(5) Let S be a "wild" embedded 2-sphere in S^3 (i.e., $\pi_1(S^3 - S, x_0)$ is non-trivial for some choice of x_0). Compute $H_1(S^3 - S)$.

(6) Suppose that X is a simply connected, compact, 5-dimensional manifold such that $H^3(X)$ is finite. Show that $H^2(X)$ is finite.

(7) Give (with proof) an example of a closed but not exact 1-form on $\mathbb{R}^2 - \{0\}$.

(8) Show that $\mathrm{SL}(n, \mathbb{R})$ is a smoothly embedded submanifold of $\mathbb{R}^{n^2}$ and compute its dimension.