Topology Qualifying Exam, August 2012

Instructions: Do all problems. **Policy on misprints:** If you feel that a problem has been misstated, then restate it in the way you believe it should be stated and solve the restated problem. Do not restate the problem so as to make the problem trivial.

1. (a.) Give a careful definition of the tangent bundle of a manifold X.
   (b.) Show that the total space of the tangent bundle of S^2 is not diffeomorphic to $S^2 \times \mathbb{R}^2$.

2. Suppose M is a smooth n–manifold, and $f\colon M \to \mathbb{R}^{2n+1}$ is injective. Let B be a closed subset of M, let U be an open subset containing B, and ϕ_B a smooth bump function for B supported in U. For any $b \in \mathbb{R}^{2n+1}$, define $g_b\colon M \to \mathbb{R}^{2n+1}$ by
$$g_b(x) = f(x) + b\,\phi_B(x).$$
   Show that there is some $b \neq 0$ in $\mathbb{R}^{2n+1}$ so that g_b is injective.

3. Let θ and η be smooth 3–forms on the 7–dimensional sphere S^7. Prove:
$$\int_{S^7} d\theta \wedge \eta = \int_{S^7} \theta \wedge d\eta$$

4. Let $f\colon S^1 \to S^1$ be given by $f(z) = z^2$, where we regard S^1 as the unit circle in $\mathbb{C}$. Find a presentation of the fundamental group of the mapping torus of f:
$$M_f = S^1 \times I/(z,1) \sim (f(z),0)$$

5. A knot K is a (smoothly) embedded copy of S^1 in S^3. For a given knot K, the knot complement is the space $S^3 \setminus K$. Use the Mayer–Vietoris sequence to compute the reduced homology groups of any knot complement. [Hint: One way to do this is to write K as the union of two embedded copies of the interval $(0,1)$. Take A to be S^3 minus one of these, B to be S^3 minus the other.]

6. Let M be a closed connected 5–manifold such that $\pi_1(M) \cong \mathbb{Z}/7\mathbb{Z}$. If $H_2(M;\mathbb{Z}) \cong \mathbb{Z}$, compute all other homology and cohomology groups of M with integral coefficients. State all theorems that you use.