Topology Qualifying Exam, January 2012

Instructions: Do all problems. **Policy on misprints:** If you feel that a problem has been misstated, then restate it in the way you believe it should be stated and solve the restated problem. Do not restate the problem so as to make the problem trivial.

1. Let M be a smooth manifold, TM its tangent bundle, and $\pi\colon TM \to M$ the bundle map. Prove the extension lemma for vector fields:

Let Y be a vector field defined on a closed subset $A \subset M$ (so $Y\colon A \to TM$ is a map satisfying $\pi \circ Y = Id_A$ and for all $p \in A$, there exists a neighborhood V_p of p in M and a smooth vector field $\widetilde{Y}$ on V_p that agrees with Y on $V_p \cap A$). If U is an open set containing A, show there exists a smooth vector field $\widetilde{Y}$ on all of M such that $\widetilde{Y}|_A = Y$ and the support of $\widetilde{Y}$ is contained in U.

2. Prove that for any 2–manifold M smoothly embedded in $\mathbb{R}^6$, there exists $v \in S^5$ such that orthogonal projection in the direction of v gives an injective map of M to a hyperplane.

3. Let M be a smooth manifold. Suppose $\gamma_0, \gamma_1\colon [0,1] \to M$ are smooth curves that are path homotopic. For every closed 1–form ω on M, prove:

$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$$

4. Show the complement of a finite set of points in $\mathbb{R}^n$ is simply connected if $n \geq 3$.

5. Define a Δ–complex structure on a Klein bottle K and use it to compute the homology groups of K with $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients.

6. Let M be a closed, orientable n–manifold. Let F^i denote $H^i(M;\mathbb{Z})$ with torsion factored out.

(a) Prove that F^i is isomorphic to $\mathrm{Hom}(H^{n-i}(M), \mathbb{Z})$. What is the isomorphism? That is, for $\alpha \in F^i$, the isomorphism takes α to a homomorphism sending $\phi \in H^{n-i}(M)$ to which element of $\mathbb{Z}$?

(b) If α generates a $\mathbb{Z}$–summand of $H^{n-i}(M;\mathbb{Z})$, prove there exists $\beta \in H^i(M;\mathbb{Z})$ so that the cup product $\alpha \smile \beta$ generates $H^n(M;\mathbb{Z})$.