Topology Qualifying Exam, August 2011

Instructions: Do all problems. **Policy on misprints:** If you feel that a problem has been misstated, then restate it in the way you believe it should be stated and solve the restated problem. Do not restate the problem so as to make the problem trivial.

1.(a) What is the definition of a trivial vector bundle?

(b) If M is a smooth manifold, TM its tangent bundle, and T^*M its cotangent bundle, show that T^*M is trivial if and only if TM is trivial using the definition in part (a).

2.(a) Show that the subset M of $\mathbb{R}^3$ defined by $(1 - z^2)(x^2 + y^2) = 1$ is a smooth submanifold of $\mathbb{R}^3$. What is its dimension?

(b) Let V be the vector field on $\mathbb{R}^3$ given by

$$V = z^2 x \frac{\partial}{\partial x} + z^2 y \frac{\partial}{\partial y} + z(1 - z^2)\frac{\partial}{\partial z}.$$

Show V restricts to a (tangent) vector field on M.

(c) Notice that $\phi_t(x, y, z) = (\cos tx - \sin ty, \sin tx + \cos ty, z)$ gives a one–parameter family of diffeomorphisms of M. For all t, determine the push–forward $(\phi_t)_* V$.

3.(a) Let M be an oriented smooth n–manifold, let $U \subset M$ be an oriented coordinate neighborhood, and let ω be a smooth n–form whose support lies in U. Define the integral $\int_M \omega$ and show that your definition is independent of any choice of coordinates used.

(b) Suppose ω is an n–form on S^n that is the pull–back of a closed form on $\mathbb{R}^{n+1}$. Find $\int_{S^n} \omega$.

4. State and prove the path lifting property for covering spaces.

5. For this problem, you may assume the homology and reduced homology groups of S^1 and of any contractible space.

(a) Compute the homology groups of the torus $S^1 \times S^1$.

(b) Using the Mayer–Vietoris sequence, compute the homology groups of the space obtained from a torus $S^1 \times S^1$ by attaching a Möbius band to the circle $S^1 \times \{x_0\}$.

6. Show that if a closed, orientable manifold M of dimension $2k$ has $H_{k-1}(M; \mathbb{Z})$ torsion free, then $H_k(M; \mathbb{Z})$ is also torsion free.