Topology Qualifying Exam, May 2011

Instructions: Do all problems. **Policy on misprints:** If you feel that a problem has been misstated, then restate it in the way you believe it should be stated and solve the restated problem. Do not restate the problem so as to make the problem trivial.

1. It is known that the tangent bundle TM over a smooth manifold M is itself a smooth manifold. Given that M is a smooth manifold, write down the charts for TM, and show that the transition functions for these charts are smooth.

2. Give a proof of the Constant–rank level set theorem, which states: Let M and N be smooth manifolds, and let $\Phi\colon M \to N$ be a smooth map whose derivative, or pushforward, $\Phi_*\colon T_pM \to T_{\Phi(p)}N$, has constant rank equal to k. Then for any $c \in N$, the level set $\Phi^{-1}(c) \subset M$ is a closed embedded submanifold of codimension k in M.

3. Let ω be a smooth 1–form on a connected manifold M such that $\int_C \omega = 0$ for every smooth closed curve C in M. Prove that $\omega = df$ for some function $f\colon M \to \mathbb{R}$.

4. Prove that any continuous map $f\colon \mathbb{R}P^2 \to S^1 \times S^1$ is homotopic to a constant map.

5. Assume X and Y are path–connected spaces. For $x \in X$ and $y \in Y$, recall that the space $X \vee Y$ is the quotient space

$$X \vee Y = \frac{X \coprod Y}{x = y},$$

where $\coprod$ denotes disjoint union.

(a) Suppose that x and y are deformation retracts of neighborhoods $U \subset X$ and $V \subset Y$, respectively. Prove that there are isomorphisms

$$\widetilde{H}_n(X \vee Y) \cong \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y)$$

for all $n \geq 0$.

(b) Again suppose that x and y are deformation retracts of neighborhoods $U \subset X$ and $V \subset Y$, respectively. Find $\pi_1(X \vee Y)$ in terms of the groups $\pi_1(X)$ and $\pi_1(Y)$.

6. Let M be a closed, connected 3–manifold and write $H_1(M;\mathbb{Z})$ as $\mathbb{Z}^r \oplus F$, the direct sum of a free abelian group of rank r and a finite group F. Show that $H_2(M;\mathbb{Z})$ is $\mathbb{Z}^r$ if M is orientable, and $\mathbb{Z}^{r-1} \oplus \mathbb{Z}_{2k}$ if M is non-orientable, for some $k \geq 1$. [In fact, it can be shown that $k = 1$, but you don't need to show $k = 1$ for this problem.]

If desired, in the non-orientable case you may use without proof the following fact, which follows from the Universal Coefficients Theorem for $\mathbb{Z}_2$–modules: As vector spaces over $\mathbb{Z}_2$, $\dim H_i(M;\mathbb{Z}_2) = \dim H_{3-i}(M;\mathbb{Z}_2)$.