Preliminary Examination – Topology, 2010-05-08

Instructions: Do ONE problem from each section. Some sections have only one problem. **Policy on misprints.** If you feel that a problem has been misstated, then restate it in the way you believe it should be stated and solve the restated problem. Do not restate the problem so as to make the problem trivial.

Section I. Do this problem.

Ia.) Give an example of a linearly ordered space that is not metrizable.

Section II. Do this problem.

IIa.) Suppose that A is a compact subset of a space X, that B is a compact subset of a space Y and that U is an open subset of the product space $X \times Y$ that contains $A \times B$. Prove that there are open subsets V of X and W of Y such that $A \times B \subset V \times W \subset U$. Give all details. Do not quote other theorems.

Section III. Do one of these two problems.

IIIa.) Consider the subset X of the Euclidean plane $\mathbb{R}^2$ that is the union of all of the following closed intervals:

$\{(1/n) \times [0,1] \,|\, n = 1, \ldots, \infty\}$

$\{[-1,0] \times (1/n) \,|\, n = 1, \ldots, \infty\}$

$\{(-1/n) \times [-1,0] \,|\, n = 1, \ldots, \infty\}$

$\{[0,1] \times (-1/n) \,|\, n = 1, \ldots, \infty\}$.

Prove that X is connected.

IIIb.) Suppose that $X_1 \supset X_2 \supset X_3 \supset \cdots$ is a nested sequence of nonempty compact connected subsets of $\mathbb{R}^n$. Prove that the intersection $X = \cap_{i=1}^{\infty} X_i$ is nonempty, compact, and connected

Section IV. Do one of these two problems.

IVa.) Let X denote a countable compact metric space. Prove that there is a 1-point subset $\{p\}$ of X that is open in X.

IVb.) For a space Y, define the derived subset $Y^{(1)}$ of Y to be the collection of points in Y that are limit points in Y. For an ordinal number α that has an immediate predecessor β, define $Y^{(\alpha)}$ to be derived set of $Y^{(\beta)}$. If α has no immediate predecessor, define $Y^{(\alpha)}$ to be the intersection of the derived sets $Y^{(\beta)}$ for $\beta < \alpha$. Use transfinite induction to prove that there is an ordinal α such that $Y^{(\alpha)} = Y^{(\alpha+1)}$.

Section V. Do one of these two problems.

Va.) Use the Mayer-Vietoris sequence and induction on dimension to calculate the reduced homology of the sphere $\mathbb{S}^n$. Use this result to prove the no retraction theorem.

Vb.) Prove that every map from the n-ball $\mathbb{B}^n$ to itself has a fixed point. You may assume the no retraction theorem.

Section VI. Do one of these two problems.

VIa.) State the Seifert-van Kampen Theorem for a space that is the union of two intersecting path connected open subsets whose intersection is path connected. Use that theorem to calculate the fundamental group of the wedge $X = T^2 \vee \mathbb{S}^1$ of the torus T^2 and the circle $\mathbb{S}^1$. Pay attention to the fact that neither T^2 nor S^1 is an open subset of X.

VIb.) Use the Seifert-van Kampen theorem to calculate a presentation for the fundamental group of a closed orientable surface of genus 3.

Section VII. Do one of these three problems.

VIIa.) Consider the wedge $X = \mathbb{S}^1 \vee \mathbb{S}^1$ of two circles. Construct a finite sheeted covering space of X that demonstrates that the fundamental group of X is not Abelian (3 sheets suffice). State the properties of covering spaces that permit you to make this conclusion that $\pi_1(X)$ is non Abelian.

VIIb.) Build the universal covering space of the wedge $X = \mathbb{S}^1 \vee \mathbb{S}^2$ of the circle $\mathbb{S}^1$ and the 2-sphere $\mathbb{S}^2$. Without giving full details, explain why this shows that the second homotopy group $\pi_2(X)$ is not finitely generated.

VIIc.) Prove that every subgroup of a finite rank free group is a free group. (Use properties of the covering space theory of a finite graph.) You may use the fact that the fundamental group of a finite bouquet of circles is a free group.