Math 113 (Calculus II)
Final Exam Form A KEY

Multiple Choice. In the grid below fill in the correct answer to each question.

1. Here is a series $\sum_{k=1}^{\infty} \frac{(-1)^n}{n(n+1)}$. Which of the following is true?

   a) The series diverges by the alternating series test.

   b) The series converges conditionally by the ratio test.

   c) The series converges absolutely by the ratio test.

   d) The series diverges by the integral test but converges by the ratio test.

   e) The series converges conditionally by a limit comparison test.

   f) The series converges absolutely by a limit comparison test.

   g) The series neither diverges nor converges.

   h) None of the above.

   Solution: f)

2. Find the first four terms of the binomial series for $(1+x)^{1/3}$.

   a) $1 + x - \frac{1}{3}x^2 + \frac{1}{81}x^3$ b) $1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3$ c) $1 + \frac{1}{6}x + \frac{1}{9}x^2 + \frac{2}{81}x^3$

   d) $1 + \frac{1}{3}x + \frac{1}{4}x^2 + \frac{1}{5}x^3$ e) None of the above.

   Solution: b)

3. Find the interval of convergence of the power series

$$\sum_{k=1}^{\infty} (-1)^k \, 2^k \frac{x^k}{k}.$$

   a) $[-1, 1]$ b) $[-2, 2]$ c) $(-2, 2]$

   d) $(-\frac{1}{2}, \frac{1}{2}]$ e) $[-\frac{1}{2}, \frac{1}{2})$ f) $(-1, 1]$

   g) The series converges for all values of x.   h) None of the above.

Solution: d)

4. The length of the curve $y = \frac{2}{3}x^{3/2}$ for $x \in [0, 2]$ is

a) $\frac{5}{3}\sqrt{5} - \frac{1}{3}$

b) $\sqrt{5} - \frac{1}{2}\ln\left(\sqrt{5} - 2\right)$

c) $2\sqrt{3} - \frac{2}{3}$

d) $\frac{16}{3} - \frac{4}{3}\sqrt{2}$

e) None of the above.

Solution: c)

5. The integral $\int_1^e \frac{\ln(x)}{x^2} dx$ equals

a) $\frac{1}{2}$

b) $-e^{-1}\ln 2 - 2e^{-1} + \ln 2 + 1$

c) $-2e^{-1} + 1$

d) $-4e^{-1} + 1$

e) None of the above.

Solution: c)

6. The ellipse $\frac{x^2}{64} + \frac{y^2}{100} = 1$ has the parametric equations

a) $x = 10\sin t, y = 8\cos t, 0 \leq t \leq 2\pi$

b) $x = 8\cos t, y = 10\sin t, 0 \leq t \leq 2\pi$

c) $x = 8\cos t, y = 10\sin t, 0 \leq t \leq \pi$

d) $x = 10\sin t, y = 8\cos t, 0 \leq t \leq \pi$

e) None of the above.

Solution: b)

7. Find $\int_0^1 x^3 \sin(x^2)\, dx$

a) $\frac{1}{2}\sin(1) - \frac{1}{2}\cos(1)$

b) $-\frac{1}{4} - \frac{3}{4}\cos(1) + \frac{1}{2}\sin(1)$

c) $-\frac{1}{2}\cos(1) + \frac{1}{2}$

d) $-\frac{1}{4}\cos 2 + \frac{1}{8}\sin 2$

e) None of the above.

Solution: a)

8. Find $\int_1^2 x^3 \ln(x)\, dx$

a) $3\ln 2 - \frac{15}{16}$

b) $3\ln 2 + \frac{5}{6}$

c) $4\ln 2 - \frac{15}{16}$

d) $4\ln 2 + 3$

e) None of the above.
Solution: $4\ln 2 - \frac{15}{16}$

Solution: c)

Short Answer. Fill in the blank with the appropriate answer.

9. (10 points)

a. What is $\lim_{n\to\infty} \frac{\ln\left(1+n^2\right)}{n}$? $\mathbf{0}$

b. Find the first 4 terms of the power series of e^{x^2} centered at 0. $\mathbf{1+x^2+\frac{x^4}{2}+\frac{x^6}{6}}$

c. Let $f(x) = x^2+1$. Find the power series of f centered at 1. $\mathbf{2+2(x-1)+(x-1)^2}$

d. What number equals $\sum_{k=0}^{\infty} \frac{1}{3^k}$? $\mathbf{\frac{3}{2}}$

e. Find $\lim_{n\to\infty} \frac{n^3-67}{n^3}$ $\mathbf{1}$

f. Identify $\int \sec(x)\,dx$. $\mathbf{\frac{1}{2}\sec x\tan x+\frac{1}{2}\ln|\sec x+\tan x|+C}$

g. What is the correct substitution to use in computing the integral, $\int_0^1 \sqrt{1-x^2}dx$? $\mathbf{x}=\sin\theta$

h. Find the antiderivative, $\int x\cos(x)\,dx$.$\mathbf{x\sin x+\cos x+C}$

i. What is the formula for the arc length of the graph of the function $y = f(x)$ for

$x \in [a,b]$? $\mathbf{\int_a^b \sqrt{1+(f'(x))^2}\,dx}$

j. In the integral $\int_0^1 \left(1+x^2\right)^{1/2} dx$ the substitution, $u = 1+x^2$ is used. Write the integral

which results. Do not try to work the integral. $\mathbf{\int_1^2 \frac{\sqrt{u}}{2\sqrt{u-1}}\,du}$

Free Response. For problems 10 - 17, write your answers in the space provided. Use the back of the page if needed, indicating that fact. Neatly show all work.

10. (7 points) Determine whether the following series converges and explain your answer.

$$\sum_{n=1}^{\infty} \frac{\sqrt[3]{n^2+1}}{3n^2+2}$$

Solution:
Form A:
For large n, the terms act like $\dfrac{n^{2/3}}{n^2} = \dfrac{1}{n^{4/3}}$, so, we will use the Limit Comparison test in comparison with $\sum \dfrac{1}{n^{4/3}}$, which converges.

$$\lim_{n\to\infty} \frac{\frac{\sqrt[3]{n^2+1}}{3n^2+2}}{\frac{1}{n^{4/3}}} = \lim_{n\to\infty} \frac{n^{4/3}\sqrt[3]{n^2+1}}{3n^2+2}$$

$$= \lim_{n\to\infty} \frac{\sqrt[3]{n^6+n^4}}{3n^2+2} = \lim_{n\to\infty} \frac{\sqrt[3]{n^6+n^4}\cdot\frac{1}{n^2}}{(3n^2+2)\cdot\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{\sqrt[3]{1+\frac{1}{n^2}}}{3+\frac{2}{n^2}} = \frac{1}{3}.$$

Form B:
For large n, the terms act like $\dfrac{n^{1/3}}{n^2} = \dfrac{1}{n^{5/3}}$, so, we will use the Limit Comparison test in comparison with $\sum \dfrac{1}{n^{5/3}}$, which converges.

$$\lim_{n\to\infty} \frac{\frac{\sqrt[3]{n+1}}{4n^2+n}}{\frac{1}{n^{5/3}}} = \lim_{n\to\infty} \frac{n^{5/3}\sqrt[3]{n+1}}{4n^2+n}$$

$$= \lim_{n\to\infty} \frac{\sqrt[3]{n^6+n^5}}{4n^2+n} = \lim_{n\to\infty} \frac{\sqrt[3]{n^6+n^5}\cdot\frac{1}{n^2}}{(4n^2+n)\cdot\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{\sqrt[3]{1+\frac{1}{n}}}{4+\frac{1}{n}} = \frac{1}{4}.$$

11. (10 points) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(x+3)^n$$

Solution:
Form A:
Using the ratio test, we see that

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(n+1)(x+3)^{n+1}}{4^{n+1}}}{\frac{(-1)^n(n)(x+3)^n}{4^n}}\right| = \lim_{n\to\infty}\frac{n+1}{4n}|x+3| = \frac{1}{4}|x+3| < 1.$$

So,

$$|x+3| < 4$$

or

$$-4 < x+3 < 4.$$

This leads to $-7 < x < 1$.
We need to check the end points. If we plug -7 into the above series we have $\sum n$, which diverges by the divergence test. If we plug 1 into the series, we have $\sum(-1)^n n$ which diverges for the same reason. The interval of convergence therefore is $(-7, 1)$.
Form B:
Using the ratio test, we see that

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(n+1)(x-2)^{n+1}}{3^{n+1}}}{\frac{(-1)^n(n)(x-2)^n}{3^n}}\right| = \lim_{n\to\infty}\frac{n+1}{3n}|x-2| = \frac{1}{3}|x-2| < 1.$$

So,

$$|x-2| < 3$$

or

$$-3 < x-2 < 3.$$

This leads to $-1 < x < 5$.
We need to check the end points. If we plug -1 into the above series we have $\sum n$, which diverges by the divergence test. If we plug 5 into the series, we have $\sum(-1)^n n$ which diverges for the same reason. The interval of convergence therefore is $(-1, 5)$.

12. (7 points) The graph of $y = x^2$ for $x \in [0, 2]$ is rotated about the y axis to form a tank that is filled with water, (64 pounds per cubit foot). Find the amount of work required to siphon all the water to the top of the tank.

Solution: If we set $y = 0$ at the bottom of the tank, and slice the tank at height y, then the radius of the cross section is $\sqrt{y}$ and the distance to move the water is $4 - y$. Hence, the work is given by

$$64\int_0^4 \pi y\,(4-y)\;dy = \frac{2048}{3}\pi.$$

13. Find

(a) (7 points) $\int \frac{dx}{x^2-3x+2}$
Solution:
Form A:

$$\frac{1}{x^2-3x+2} = \frac{1}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}.$$
$$1 = A(x-1) + B(x-2).$$

If $x = 1$, we have $B = -1$. If $x = 2$, we have $A = 1$. Thus,

$$\int \frac{dx}{x^2-3x+2} = \int (\frac{1}{x-2} - \frac{1}{x-1})\, dx = \ln|x-2| - \ln|x-1| + C.$$

Form B:

$$\frac{1}{x^2-4x-5} = \frac{1}{(x-5)(x+1)} = \frac{A}{x-5} + \frac{B}{x+1}.$$
$$1 = A(x+1) + B(x-5).$$

If $x = -1$, we have $B = -\frac{1}{6}$. If $x = 5$, we have $A = \frac{1}{6}$. Thus,

$$\int \frac{dx}{x^2-4x-5} = \frac{1}{6}\int (\frac{1}{x-5} - \frac{1}{x+1})\, dx = \frac{1}{6}(\ln|x-5| - \ln|x+1|) + C.$$

(b) (7 points) $\int x\sqrt{2x-3}dx$
Solution: Let $u = 2x-3$. Then, $du = 2\,dx$, and $x = (u+3)/2$. The integral becomes

$$\int \frac{1}{2}(u+3)u^{1/2}\,\frac{du}{2} = \frac{1}{4}\int u^{3/2} + 3u^{1/2})\,du$$
$$= \frac{1}{4}(\frac{2}{5}u^{5/2} + 3\cdot\frac{2}{3}u^{3/2}) + C$$
$$= \frac{1}{10}(2x-3)^{5/2} + \frac{1}{2}(2x-3)^{3/2} + C.$$

14. (7 points) The region between $y = \ln x$ which lies between $x = 1$, $x = 2$, and the x axis is revolved about the line $x = -1$. Find the volume of the resulting solid of revolution.

Solution: $2\pi \int_1^2 (x+1)\ln(x)\,dx$
Use integration by parts: Let $u = \ln x$, so $du = 1/x\,dx$, $dv = x+1$ and $v = \frac{x^2}{2} + x$. The above is therefore equal to

$$2\pi((\frac{x^2}{2} + x)\ln x - \int \frac{x}{2} + 1\,dx) = 2\pi((\frac{x^2}{2} + x)\ln x - \frac{x^2}{4} - x)|_1^2$$
$$= 2\pi\left(4\ln 2 - \frac{7}{4}\right)$$

15. (7 points) The base of a solid is the inside of the circle, $x^2+y^2 \leq 9$. Cross sections perpendicular to the x axis are squares. Find the volume of the resulting solid.

Solution:
Form A:

$$\int_{-3}^{3} \left(2\sqrt{9-x^2}\right)^2 dx = \int_{-3}^{3} 36 - 4x^2\, dx = 144.$$

Form B:

$$\int_{-2}^{2} \left(2\sqrt{4-x^2}\right)^2 dx = \int_{-2}^{2} 16 - 4x^2\, dx = \frac{128}{3}.$$

16. (7 points) Determine whether the integral $\int_0^1 \frac{\sin(x)}{\sqrt{1-x^2}} dx$ converges.

Solution: Notice that on $0 \leq x \leq 1$, $0 \leq \sin x \leq 1$. Thus,

$$\frac{\sin(x)}{\sqrt{1-x^2}} < \frac{1}{\sqrt{1-x^2}}.$$

Since

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx = \lim_{b\to 1^-} \int_0^b \frac{1}{\sqrt{1-x^2}}\, dx$$
$$= \lim_{b\to 1^-} \sin^{-1}(x)|0^b = \lim_{b\to 1^-} \sin^{-1} b = \frac{\pi}{2}.$$

Since this integral converges, the above does as well by the comparison test.

17. (7 points) Find $\int_1^\infty \frac{1}{x(x^2+1)} dx$

Solution:

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$
$$1 = A(x^2+1) + (Bx+C)x.$$

If $x=0$, we see that $A=1$. If $x=-1$, we have $B-C=-1$ and if $x=1$, we have $B+C=-1$. Thus, $B=-1$ and $C=0$.

$$\int_1^\infty \frac{1}{x(x^2+1)} dx = \lim_{b\to\infty} \int_1^b \frac{1}{x(x^2+1)} dx$$
$$= \lim_{b\to\infty} \int_1^b \frac{1}{x} - \frac{x}{x^2+1}\, dx = \lim_{b\to\infty} (\ln x - \frac{1}{2}\ln(x^2+1))|_1^b$$
$$= \lim_{b\to\infty} (\ln b - \ln\sqrt{b^2+1} + \frac{1}{2}\ln 2) = \lim_{b\to\infty} \ln\left(\frac{b}{\sqrt{b^2+1}}\right) + \frac{1}{2}\ln 2 = \frac{1}{2}\ln 2$$

END OF EXAM