Math 113 (Calculus II)
Final Exam   Form A KEY

Multiple Choice. Fill in the answer to each problem on your computer-scored answer sheet. Make sure your name, section and instructor are on that sheet.

1. Find the area of the region bounded by $y = e^{2x}$, $y = e^{x}$, and $x = \ln 3$.

   a) 6 b) 2 c) 0 d) $\frac{1}{2}$

   e) $\frac{3}{2} - \frac{1}{2}e^2 + e$ f) $\frac{3}{2}$ g) 13 h) 4

   i) None of these.

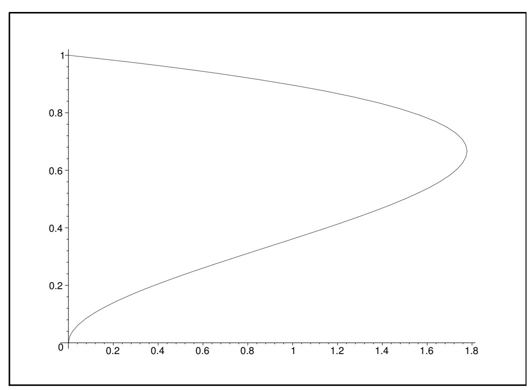
2. Find the volume of the solid obtained by rotating about the x-axis the region in the first quadrant that is enclosed by the curves $y = \cos x$, $y = \sin x$, and $x = 0$.

   a) $\frac{1}{2}$ b) $\frac{\pi}{4}$ c) $\frac{\sqrt{2}\pi}{4}$ d) π

   e) $\frac{\pi}{2}$ f) 2π g) 2 h) 0

   i) None of these.

3. The curve $x = 12(y^2 - y^3)$ and the y-axis enclose a region in the first quadrant (see figure). Find the volume of the solid obtained by rotating this region about the x-axis.



   a) $\frac{3\pi}{5}$ b) 2π c) $\frac{6\pi}{5}$ d) $\frac{\pi}{20}$

   e) 24π f) 12π g) $\frac{\pi}{10}$ h) $\frac{9\pi}{5}$

   i) None of the above

4. What is the value of the integral $\displaystyle\int_0^2 \frac{dx}{(1-x)^2}$?

a) -2 b) -1 c) 2

d) $\tan^{-1} 2$ e) $\ln 2$ f) 0

g) $\frac{1}{2}\ln 3$ h) The integral diverges. i) None of these

5. Find the radius of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x+2)^n}{n2^n}$.

a) 1 b) 0 c) 4 d) ∞

e) $\frac{1}{2}$ f) 2 g) 6 h) None of these.

6. What is the third non-zero term in the Maclaurin series of $f(x) = \dfrac{1}{\sqrt{x+4}}$?

a) $\dfrac{x^2}{4}$ b) $\dfrac{3x^2}{128}$ c) $-\dfrac{x^2}{64}$ d) $12x^2$

e) $\dfrac{3x^2}{256}$ f) $40x^2$ g) $\dfrac{15x^2}{128}$ h) None of these.

7. Find the area of the region enclosed by one loop of the curve $r = 4 + 2\cos\theta$.

a) 16π b) 9π c) 18π d) 36π

e) 11π f) 32π g) 8π h) None of these

8. Find a power series representation for $f(x) = \dfrac{1}{(1+2x)^2}$.

a) $\displaystyle\sum_{n=0}^{\infty}(-1)^n 2^n x^n$ b) $\displaystyle\sum_{n=0}^{\infty}(n+1)2^n x^n$ c) $\displaystyle\sum_{n=0}^{\infty} 2^{2n} x^{2n}$

d) $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1} 2^{n-1} x^n$ e) $\displaystyle\sum_{n=1}^{\infty}(-1)^n n 2^n x^{n-1}$ f) $\displaystyle\sum_{n=0}^{\infty}(-1)^n (n+1) 2^n x^n$

g) None of the above

9. Approximate $x^2 \cos 3x$ near $x = 0$ with a sixth degree Taylor polynomial.

a) $1 - \dfrac{9x^2}{2} + \dfrac{81x^4}{24} - \dfrac{729x^6}{720}$ b) $1 + \dfrac{9x^2}{2} + \dfrac{81x^4}{24} + \dfrac{729x^6}{720}$ c) $x^2 - \dfrac{9x^4}{2} + \dfrac{81x^6}{4}$

d) $x^2 + \dfrac{9x^4}{2} + \dfrac{81x^6}{4}$ e) $x^2 - \dfrac{9x^4}{2} + \dfrac{81x^6}{24}$ f) $3x^2 - \dfrac{9x^5}{2}$

g) $1 - 3x^2 + \dfrac{9x^5}{2}$ h) None of the above

Free response: Write your answer in the space provided.

In problems 10 to 12, evaluate the following indefinite or definite integrals.

10. (7 points) $\int \frac{x^2}{(4x^2+9)^2}\,dx$.

Solution:

Form A:

$x = \frac{3}{2}\tan\theta,\ dx = \frac{3}{2}\sec^2\theta\,d\theta.$

$$\begin{aligned}
\int \frac{x^2}{(4x^2+9)^2}\,dx &= \int \frac{\frac{9}{4}\tan^2\theta}{(9\tan^2\theta+9)^2}\frac{3}{2}\sec^2\theta\,d\theta \\
&= \frac{1}{24}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta = \frac{1}{24}\int \frac{\sec^2\theta - 1}{\sec^2\theta}\,d\theta \\
&= \frac{1}{24}\int (1-\cos^2\theta)\,d\theta = \frac{1}{24}\int (\frac{1}{2}-\frac{1}{2}\cos(2\theta))\,d\theta \\
&= \frac{1}{48}\theta - \frac{1}{48}\sin(2\theta) + C = \frac{1}{48}\theta - \frac{1}{48}\sin\theta\cos\theta + C \\
&= \frac{1}{48}\tan^{-1}(\frac{2x}{3}) - \frac{1}{8}\frac{x}{x^2+9} + C
\end{aligned}$$

Form B: $x = \frac{5}{3}\tan\theta,\ dx = \frac{5}{3}\sec^2\theta\,d\theta.$

$$\int \frac{x^2}{(9x^2+25)^2}\,d\theta = \frac{1}{270}\tan^{-1}(\frac{3}{5}x) - \frac{1}{18}\frac{x}{9x^2+25} + C$$

11. (7 points) $\int_0^{\pi} x\sin x\,dx$

Solution: Use integration by parts. $u = x$, $du = dx$, $dv = \sin x\,dx$, $v = -\cos x$.

$$\int_0^{\pi} x\sin x\,dx = (-x\cos x + \int \cos x\,dx)|_0^{\pi} = (-x\cos x + \sin x)|_0^{\pi} = \pi.$$

12. (7 points) $\displaystyle\int \frac{dx}{(x+1)(x^2+1)}$

Solution:

Form A:

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$
$$1 = A(x^2+1) + (Bx+C)(x+1)$$
$$A = 1/2,\ B = -1/2,\ C = 1/2$$

$$\begin{aligned}\int \frac{dx}{(x+1)(x^2+1)} &= \frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{x}{x^2+1}\,dx + \frac{1}{2}\int \frac{dx}{x^2+1} \\ &= \frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{2x}{x^2+1}\,dx + \frac{1}{2}\int \frac{dx}{x^2+1}.\end{aligned}$$

Let $u = x^2+1$, $du = 2x\,dx$. The above becomes

$$\frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{1}{u}\,du + +\frac{1}{2}\int \frac{dx}{x^2+1}$$

$$= \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x^2+4) + \frac{1}{2}\tan^{-1}(x) + C.$$

Form B:

$$\frac{1}{(x-2)(x^2+1)} = \frac{1}{5}\frac{1}{x-2} - \frac{1}{5}\frac{x}{x^2+1} - \frac{2}{5}\frac{1}{x^2+1}$$
$$\begin{aligned}\int \frac{dx}{(x-2)(x^2+1)} &= \frac{1}{5}\int \frac{dx}{x-2} - \frac{1}{10}\int \frac{2x}{x^2+1}\,dx - \frac{2}{5}\int \frac{dx}{x^2+1} \\ &= \frac{1}{5}\ln|x-2| - \frac{1}{10}\ln(x^2+1) - \frac{2}{5}\tan^{-1}(x) + C\end{aligned}$$

13. (8 points) Set up but *do not evaluate* the integrals.

(a) Set up the integral used to determine the arc length of the curve defined by the equation $y^2 + 2y = 2x + 1$ from $(-1, -1)$ to $(7, 3)$.

Solution:

$$x = \frac{1}{2}y^2 + y - \frac{1}{2}, \quad x' = y + 1$$

$$s = \int_{-1}^{3} \sqrt{1 + (y+1)^2}\, dy$$

(b) Set up the integral used to find the length of the curve defined by the parametric equations

$$x = e^t - t$$
$$y = 4e^{t/2} \quad 0 \le t \le 1.$$

Solution:

Form A:

$$x' = e^t - 1, \quad y' = 2e^{t/2}$$

$$s = \int_0^1 \sqrt{(e^t - 1)^2 + 4e^t}\, dt$$

Form B:

$$x' = e^t + 2, \quad y' = 4e^t$$

$$s = \int_0^1 \sqrt{(e^t + 2)^2 + 4e^{2t}}\, dt$$

14. (7 points) Find the sum of the series, or show that the series diverges. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n}$

Solution:

Form A:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n} = \sum_{n=1}^{\infty} \left(-\frac{1}{4}\right)^n = \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n - 1$$
$$= \frac{1}{1 + \frac{1}{4}} - 1 = \frac{4}{5} - 1 = -\frac{1}{5}.$$

Form B:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n} = \sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n - 1$$
$$= \frac{1}{1 + \frac{1}{3}} - 1 = \frac{3}{4} - 1 = -\frac{1}{4}.$$

15. (12 points) For each of the following series, use an appropriate test to prove that the series converges, or to prove that it diverges. If the series is alternating and convergent, show whether it is absolutely or conditionally convergent.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln^2 n}$

Solution: We use the integral test. Let $u = \ln x$, $du = (1/x)\, dx$.

$$\int_2^{\infty} \frac{1}{x \ln^2 x}\, dx = \int_{\ln(2)}^{\infty} \frac{1}{u^2}\, du$$

Since we know the latter integral converges, the series converges by the Integral test.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n}{n^2+5}$

Solution:

Form A:

Notice that $\dfrac{n}{n^2+5}$ decreases to 0. Thus, the above series converges by the alternating series test. We need to test for absolute convergence. We will apply the limit comparison test to $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n^2+5}$ using $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ as a comparison series. Note that it does not matter that the two series start at a different place, because the first few terms will not make a series diverge. Notice that

$$\lim_{n\to\infty} \frac{\frac{n}{n^2+5}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+5} = 1.$$

Hence, the series converges conditionally.

Form B:

As above, the Alternating series test shows convergence, but the limit comparision test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ shows the convergence is conditional.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\ln n}$

Solution: Notice that $\ln n < n$, so

$$\frac{1}{1+\ln n} > \frac{1}{2\ln n} > \frac{1}{2n}.$$

Since $\displaystyle\sum \frac{1}{2n}$ diverges, the series above diverges by the comparison test.

16. (8 points) A bag of sand originally weighing 144 lb is lifted 18 ft at a constant rate. As it rises, the sand leaks out at a constant rate, and the sand is half gone by the time it had been lifted the 18 ft. How much work was done in lifting the bag of sand? (Neglect the weight of the bag and the lifting equipment).

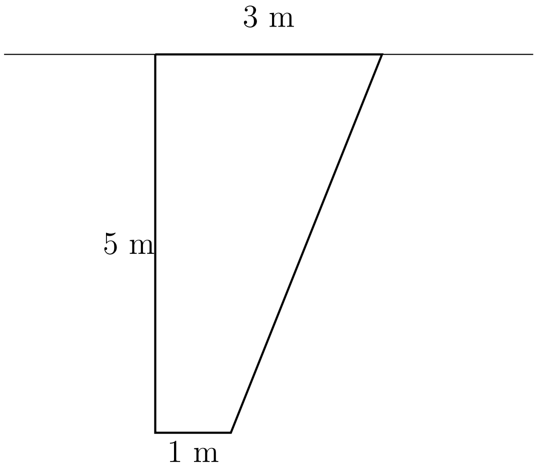
Solution: Since the sand leaks out at a constant rate, the force function is linear. At height $y = 0$, we have $F = 144$. At height $y = 18$, the force is $F = 72$. Thus, we need only find a linear function between $(0, 144)$ and $(18, 72)$. Note that the slope is

$$\frac{72 - 144}{18 - 0} = -\frac{72}{18} = -4.$$

Thus, the linear equation is $F - 144 = -4(x - 0)$, or $F(y) = -4y + 144$, and the work is

$$\int_0^{18} -4y + 144\, dy = (-2y^2 + 144y)|_0^{18} = 1944 \text{ lbs.}$$

17. (8 points) Find the force due to fluid pressure on the pictured plate that is submerged vertically in water until the top of the plate is at the top of the water. (Remember the density of water is 1000 kg/m^3.)



Solution:

Form A:

Let y represent the depth of the water and be 0 at the top of the water. At depth y, we need to know the width across the plate. Note that we can easily do this if we divide the plate into a rectangle and a right triangle. Using similar triangles, the width across the triangle portion at depth y (we'll call it l), can be found by

$$\frac{l}{2} = \frac{5-y}{5}$$

so $l = \frac{2}{5}(5-y)$ and the width across the plate is $x = 1 + \frac{2}{5}(5-y) = 3 - \frac{2}{5}y$. The force due to fluid pressure is therefore

$$F = \int_0^5 \rho g y(3 - \frac{2}{5}y)\, dy = \rho g \int_0^5 3y - \frac{2}{5}y^2|_0^5 = \rho g(\frac{3}{2}y^2 - \frac{2}{15}y^3)|_0^5$$

$$= 9800(\frac{75}{2} - \frac{50}{3}) = \frac{612500}{3} \text{ lbs.}$$

Form B:

Here the width across the plate is found by calculating

$$\frac{l}{3} = \frac{6-y}{6},\ l = \frac{1}{2}(6-y),\ x = 1 + \frac{1}{2}(6-y) = 4 - \frac{1}{2}y.$$

$$F = \int_0^5 \rho g y(4 - \frac{1}{2}y)\, dy = 352800 \text{ lbs.}$$

END OF EXAM