ENTROPY, SMOOTH ERGODIC THEORY, AND RIGIDITY OF GROUP ACTIONS

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1. INTRODUCTION

This is a preliminary version of lecture notes based on a 6 hour course given at the workshop Dynamics Beyond Uniform Hyperbolicity held in Provo, Utah June 2017.

The goal of this course is two-fold. First we will present a number of tools and results in the smooth ergodic theory of actions of higher-rank abelian groups including Lyapunov exponents, metric entropy, and the relationship between entropy, exponents, and geometry of conditional measures. We will explain the main proposition (the invariance principle that “non-resonance implies invariance”) from the work of Brown-Rodriguez Hertz-Wang and explain their main theorem: every action of a lattice in $\text{SL}(n, \mathbb{R})$ on an $(n - 2)$-dimensional manifold preserves a Borel probability measure.

Second, we will outline the proof of the main theorem of Brown-Fisher-Hurtado: every action of a cocompact lattice in $\text{SL}(n, \mathbb{R})$ on a $(n - 2)$-dimensional manifold is finite. We will explain some of the main tools used in the proof: Strong property (T), Margulis superrigidity, and Zimmer’s cocycle superrigidity and how they combine with the invariance principle from the work of Brown-Rodriguez Hertz-Wang and Ratner’s theorems to prove the theorem.

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Part 1. Lattices acting on manifolds and the Zimmer program

2. Smooth lattice actions

The main goal of the course will be to understand properties and to classify smooth actions of certain countable groups $\Gamma$ on compact manifolds.

2.1. Lattices in Lie groups. Let $G$ be a connected semisimple Lie group. Recall that semisimple Lie groups are unimodular and hence admit a bi-invariant measure, called the Haar measure, which is unique up to normalization. A lattice in $G$ is a discrete subgroup $\Gamma \subset G$ with finite co-volume. That is, if $D$ is a measurable fundamental domain for the right-action of $\Gamma$ on $G$ then $D$ has finite volume. If the quotient $G/\Gamma$ is compact, we say that $\Gamma$ is a cocompact lattice. If $G/\Gamma$ is finite volume but not cocompact we say that $\Gamma$ is non-uniform. The quotient manifold $G/\Gamma$ by the right action of $\Gamma$ then admits a left-action by $G$ and the Haar measure descends to finite $G$-invariant measure on $G/\Gamma$ which we normalize to be a probability measure.

For concreteness we will primarily consider the case that $G = \text{SL}(n, \mathbb{R})$.

Example 2.1. The primary example of a lattice in $G = \text{SL}(n, \mathbb{R})$ is $\Gamma = \text{SL}(n, \mathbb{Z})$. Note that $\text{SL}(n, \mathbb{Z})$ is not cocompact in $\text{SL}(n, \mathbb{R})$. However, $\text{SL}(n, \mathbb{R})$ does possess cocompact lattices.

Example 2.2. In the case $G = \text{SL}(2, \mathbb{R})$ the fundamental group of any hyperbolic surface is a lattice in $G$. In particular, the fundamental group of a compact hyperbolic surface is a cocompact lattice in $G$.

In particular, the free group $\Gamma = F_2$ on two generators is a lattice in $G$. This can be seen by giving the punctured torus $S = \mathbb{T}^2 \setminus \{\text{pt}\}$ a hyperbolic metric and identifying the fundamental group of $S$ with the deck group of the hyperbolic plane $\mathbb{H} = \text{SO}(2, \mathbb{R}) \backslash \text{SL}(2, \mathbb{R})$.

2.2. Rank of $G$. Every semisimple Lie group admits an Iwasawa decomposition $G = KAN$ where $K$ is compact, $A$ is a connected free abelian group, and $N$ is unipotent. The the dimension of $A$ is the rank of $G$.

In the case of $G = \text{SL}(n, \mathbb{R})$ we have

$$K = \text{SO}(n, \mathbb{R}), \quad A = \{\text{diag}(e^{t_1}, e^{t_2}, \ldots, e^{t_n}) : t_1 + \ldots + t_n = 1\},$$

and

$$N = \begin{cases} \begin{pmatrix} 1 & * & \cdots & * & * \\ 0 & 1 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & * \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \end{cases}.$$ 

Note that, as we require determinant 1, we have $\text{diag}(e^{t_1}, e^{t_2}, \ldots, e^{t_n}) \in \text{SL}(n, \mathbb{R})$ if and only if $t_1 + \ldots + t_n = 0$. Thus $A \simeq \mathbb{R}^{n-1}$ and the rank of $\text{SL}(n, \mathbb{R})$ is $n - 1$.

We say that an almost-simple Lie group $G$ is higher-rank if the rank is at least 2. In particular $G = \text{SL}(n, \mathbb{R})$ is higher-rank when $n \geq 3$. 
2.3. **Lattices acting on manifolds.** Let $M$ be a compact, boundaryless manifold. Recall that a $C^r$ action $\alpha$ of $\Gamma$ on $M$ is a homomorphism $\alpha: \Gamma \to \text{Diff}^r(M)$ from $\Gamma$ to the group of $C^r$-diffeomorphisms of $M$. If $\text{vol}$ is a fixed volume form on $M$ we write $\text{Diff}^r_{\text{vol}}(M)$ for the group of $C^r$-diffeomorphisms preserving the volume form $\text{vol}$. A **volume-preserving** action is a homomorphism $\alpha: \Gamma \to \text{Diff}^r_{\text{vol}}(M)$ for some volume form $\text{vol}$.

2.4. **Standard actions.** We present a number of standard examples of lattices acting on manifolds.

**Example 2.3 (Trivial actions).** Let $\Gamma'$ be a finite-index normal subgroup of $\Gamma$. Then $F = \Gamma / \Gamma'$ is finite. Suppose the finite group $F$ acts on a manifold $M$. Since $F$ is a quotient of $\Gamma$ we obtain an induced action of $\Gamma$ on $M$.

Note that an action of a finite group preserves a volume form simply by averaging any volume form by the action.

**Definition 2.4.** An action $\alpha: \Gamma \to \text{Diff}(M)$ is **trivial** if it factors through the action of a finite subgroup. That is, $\alpha$ is trivial if there is a finite-index normal subgroup $\Gamma' \subset \Gamma$ such that $\alpha|_{\Gamma'}$ is the identity.

**Example 2.5 (Affine actions).** Let $\Gamma = \text{SL}(n, \mathbb{Z})$. Let $M = \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ be the $n$-dimensional torus. We have a natural action $\alpha: \Gamma \to \text{Diff}(\mathbb{T}^n)$ given by

$$\alpha(\gamma)(x + \mathbb{Z}^n) = \gamma \cdot x + \mathbb{Z}^n$$

for any matrix $\gamma \in \text{SL}(n, \mathbb{Z})$.

To generalize this example to other lattices, let $\Gamma \subset \text{SL}(n, \mathbb{R})$ be any lattice and let $\rho: \Gamma \to \text{SL}(d, \mathbb{Z})$ be any representation. Then we have a natural action $\alpha: \Gamma \to \text{Diff}(\mathbb{T}^d)$ given as

$$\alpha(\gamma)(x + \mathbb{Z}^d) = \rho(\gamma) \cdot x + \mathbb{Z}^d.$$

Note that these examples preserve a volume form, namely, the Lebesgue measure on $\mathbb{T}^d$.

**Example 2.6 (Projective actions).** Let $\Gamma \subset \text{SL}(n, \mathbb{R})$ be any lattice. Then $\Gamma$ acts naturally on $\mathbb{R}^n$ by matrix multiplication. Moreover, action of $\Gamma$ on $\mathbb{R}^n$ induces an action of $\Gamma$ on the sphere $S^{n-1}$ thought of as the set of unit vectors in $\mathbb{R}^n$: we have $\alpha: \Gamma \to \text{Diff}(S^n)$ given by

$$\alpha(\gamma)(x) = \frac{\gamma \cdot x}{\|\gamma \cdot x\|}.$$

Alternatively we could act on the space of lines in $\mathbb{R}^n$ and obtain of $\Gamma$ an action on the $(n-1)$-dimensional projective space. This action does not preserve a volume; in fact there is no invariant probability measure for this action.

**Remark 2.7 (Actions on boundaries).** Example 2.6 generalizes to actions of lattices $\Gamma$ in $G$ acting on boundaries of $G$. Given a semisimple Lie group $G$, a parabolic subgroup $Q \subset G$, and a lattice $\Gamma \subset G$, the coset space $M = G/Q$ is compact and $\Gamma$ acts on $M$ naturally as

$$\alpha(\gamma)(xQ) = \gamma xQ.$$

These actions never preserve a volume form.

\[^1\text{From a theorem of Margulis [Mar1] we know that all normal subgroups of higher-rank lattices are either finite or of finite-index.}\]
2.5. **Non-standard actions.** For lattices in rank-one Lie groups such as \( SL(2, \mathbb{R}) \), there picture is very different from the previous subsection. There exist non-trivial actions that are non-algebraic; that is, there exists actions that are not (semi-)conjugate to an algebraic action.

**Example 2.8 (Actions of free groups).** Let \( G = SL(2, \mathbb{Z}) \). We’ve seen that the free group \( \Gamma = F_2 \) is a lattice in \( G \) (thought of as the fundamental group of the punctured torus). Let \( M \) be any manifold and let \( f, g \in \text{Diff}(M) \). Then \( f \) and \( g \) generate an action of \( \Gamma \) on \( M \) that is not of an algebraic origin. In particular, there is no expectation that any rigidity phenomena hold for actions of lattices in \( SL(2, \mathbb{R}) \).

**Example 2.9 (Non-standard Anosov actions of \( SL(2, \mathbb{Z}) \)).** Consider the standard action \( \alpha_0 \) of \( SL(2, \mathbb{Z}) \) on the 2 torus \( T^2 \) constructed as in Example 2.5. In [Hur1, Example 7.21], Hurder presents an example of a 1-parameter family of deformations \( \alpha_t : SL(n, \mathbb{Z}) \to \text{Diff}(T^2) \) of \( \alpha_0 \) with the following properties:

1. Each \( \alpha_t \) is a real-analytic, volume preserving action
2. For \( t > 0 \), \( \alpha_t \) is not topologically conjugate to \( \alpha_0 \), (even when restricted to a finite index subgroup of \( SL(2, \mathbb{Z}) \)).

Moreover, since \( \alpha_0 \) is an Anosov action and since the Anosov property is an open property we have that

3. each \( \alpha_t \) is an Anosov action.

This shows that even Anosov actions of \( SL(2, \mathbb{Z}) \) fail to exhibit local rigidity properties and that there exist “exotic” Anosov actions of \( SL(2, \mathbb{Z}) \).

One can similarly produce exotic non-volume preserving actions of \( SL(2, \mathbb{Z}) \) on the circle by deforming the standard projective action.

### 3. The Zimmer Program

3.1. **Motivating questions and main conjectures.** Considering actions of lattices in rank-1 groups, we have seen that it is easy to construct exotic actions of free groups and Example 2.9 shows there are exotic Anosov actions of \( SL(2, \mathbb{Z}) \) on tori.

However, for actions of lattices in higher-rank, almost-simple Lie groups, the main conjecture of the Zimmer program is that all non-trivial actions are built out of standard algebraic examples using standard constructions. The above examples and conjecture raise a number of more precise questions. For concreteness, fix \( n \geq 3 \) and let \( G = SL(n, \mathbb{R}) \). Let \( \Gamma \subset G \) be a lattice.

**Question 3.1.** Is there a non-trivial action of \( \Gamma \) on a manifold of dimension \( < n - 1 \)?

**Question 3.2.** Is there a non-trivial, volume-preserving action of \( \Gamma \) on a manifold of dimension \( < n \)?

Questions 3.1 and 3.2 are usually referred to as the **Zimmer conjecture.** One expects the answers to both questions to be ‘no’ as there are no algebraic examples of such actions.

**Conjecture 3.3.** For \( n \geq 3 \), let \( \Gamma \subset SL(n, \mathbb{R}) \) be a lattice. Let \( M \) be a compact manifold.

1. If \( \dim(M) < n - 1 \) then any homomorphism \( \Gamma \to \text{Diff}^2(M) \) has finite image.

2. In addition, if \( \text{vol} \) is a volume form on \( M \) and \( \dim(M) = n - 1 \) then any homomorphism \( \Gamma \to \text{Diff}^2_{\text{vol}}(M) \) has finite image.
Previous results include results when $\dim(M) = 1$ [Wit, BM, Ghy], when $\dim(M) = 2$ assuming an invariant volume (or measure) [Pol, FH], and for real-analytic and holomorphic actions [FS] and [CZ].

Recently, the first author with David Fisher and Sebastian Hurtado gave a definitive answer of ‘no’ to both of these questions in the case that $\Gamma \subset G$ is cocompact. The proof of this result will be outlined in Part 3.

A related question motivated by Conjecture 3.3 is the following.

**Question 3.4.** Does every action of $\Gamma$ on a manifold of dimension $< n - 1$ preserve a volume?

The authors together with Federico Rodriguez Hertz studied Question 3.4 and were able to show that all such actions preserve some probability measure in [BRHW2]. A key idea in [BRHW2] reappears as a key ingredient in [BFH]. We will discuss these ideas and the proof of the main Theorem of [BRHW2] in Section 10 of Part 2.

Motivated by Example 2.5 we pose the following:

**Question 3.5.** Is every non-trivial action of $\Gamma$ on an $n$-torus of the type considered in Example 2.5? What about volume-preserving actions?

**Definition 3.6.** We say an action $\alpha : \Gamma \to \text{Diff}(M)$ is Anosov if $\alpha(\gamma)$ is an Anosov diffeomorphism for some $\gamma \in \Gamma$.

Note that the example of $\text{SL}(n, \mathbb{Z})$ acting on $\mathbb{T}^n$ considered in Example 2.5 is Anosov as $\text{SL}(n, \mathbb{Z})$ contains a hyperbolic matrix. However, there exist representations $\rho : \text{SL}(n, \mathbb{Z}) \to \text{SL}(d, \mathbb{Z})$ which induce non-Anosov affine actions on $\mathbb{T}^d$.

As rigidity of Anosov actions is expected to be easier to establish we refine Question 3.5.

**Question 3.7.** Is every non-trivial Anosov action of $\Gamma$ on a torus of the type considered in Example 2.5? What about volume-preserving actions?

For Anosov actions on tori, there are a number of results showing that all measure-preserving Anosov actions are of the type considered in Example 2.5. See in particular [FL, GS, KLZ, MQ]. See also [Hur1] for results on deformation rigidity, [Hur1, Hur2, Lew, Qia2] for results on infinitesimal rigidity, and [Hur1, QY, KL, KLZ, GS, Qia1] for results on local rigidity of Anosov actions given various hypotheses.

Recently, [BRHW1] gave a new mechanism to study rigidity of Anosov actions on tori; in particular, it is shown in [BRHW1] that all Anosov actions (satisfying a certain lifting conditions which holds, for instance, when the lattice is cocompact) of higher-rank lattices are of the type considered in Example 2.5, even when the action is not assumed to preserve a measure.

One motivation for the above and related questions in Zimmer program comes from a large collection of rigidity results for linear representations $\rho : \Gamma \to \text{SL}(d, \mathbb{R})$ of lattices $\Gamma$ in higher-rank Lie groups. For instance Margulis’s superrigidity theorem greatly restricts the possible linear representations $\rho : \Gamma \to \text{SL}(d, \mathbb{R})$; every such representation $\rho : \Gamma \to \text{SL}(d, \mathbb{R})$ is, up to a compact error, the restriction to $\Gamma$ of a representation $\tilde{\rho} : \text{SL}(n, \mathbb{Z}) \to \text{SL}(d, \mathbb{R})$.

2For instance, the adjoint representation $\rho : \text{SL}(n, \mathbb{Z}) \to \text{SL}(n^2 - 1, \mathbb{Z})$ given by identifying $\mathbb{R}^n \otimes \mathbb{Z}$ with $\mathfrak{sl}(n, \mathbb{R})$ and defining for $g \in \text{SL}(n, \mathbb{Z})$ and $X \in \mathfrak{sl}(n, \mathbb{R})$ 

$$\rho(g)(X) = g X g^{-1}.$$
\( \hat{\rho} : G \to \mathrm{SL}(d, \mathbb{R}) \). One motivating philosophy behind the Zimmer program is to generalize rigidity phenomena known to hold for linear representations \( \rho : \Gamma \to \mathrm{SL}(d, \mathbb{R}) \) to non-linear representations \( \alpha : \Gamma \to \mathrm{Diff}(M) \). In particular, the expectation is that, when \( \Gamma \) is a lattice in a higher-rank simple Lie group, all non-linear representations \( \alpha : \Gamma \to \mathrm{Diff}(M) \) are constructed from algebraic examples via standard constructions. We refer more to the surveys [Fis1, Fis2] for more details.

3.2. Two recent results in the Zimmer program. The following two recent theorems address Questions 3.1–3.2 above. In the next section, we will outline their proofs.

**Theorem 3.8** ([BRHW2, Theorem 1.6]). For \( n \geq 3 \), let \( \Gamma \subset \mathrm{SL}(n, \mathbb{R}) \) be a lattice. Let \( M \) be a manifold with \( \dim(M) < n \). Then, for any \( C^{1+\beta} \) action \( \alpha : \Gamma \to \mathrm{Diff}^{1+\beta}(M) \), there exists an \( \alpha \)-invariant Borel probability measure.

**Theorem 3.9** ([BFH, Theorem 1.1]). For \( n \geq 3 \), let \( \Gamma \subset \mathrm{SL}(n, \mathbb{R}) \) be a cocompact lattice. Let \( M \) be a compact manifold.

1. If \( \dim(M) < n - 1 \) then any homomorphism \( \Gamma \to \mathrm{Diff}^2(M) \) has finite image.
2. In addition, if \( \text{vol} \) is a volume form on \( M \) and \( \dim(M) = n - 1 \) then any homomorphism \( \Gamma \to \mathrm{Diff}^2_{\text{vol}}(M) \) has finite image.

4. Cocycle superrigidity and early evidence for Conjecture 3.3

The original conjecture posed by Zimmer was Conjecture 3.3(2) (see for example [Zim2, Conjecture II]). Conjecture 3.3(1) was formulated later and first appears in print in [FS, Conjecture I]. The reason Zimmer posed his conjecture as Conjecture 3.3(2) is that the strongest evidence for the most general form of the conjecture—Zimmer’s cocycle superrigidity theorem—requires an invariant measure for the group action. Zimmer’s cocycle superrigidity also provides strong evidence for conjectures related to Questions 3.5 and 3.7 as well as other conjectures in the Zimmer program and is typically used in proofs of all partial results towards solving such conjectures.

4.1. Cocycles over group actions. Consider a standard probability space \((X, \mu)\). Let \( G \) be a locally compact topological group and let \( \alpha : G \to \text{Aut}(X, \mu) \) be an action of \( G \) by measurable \( \mu \)-preserving transformations. A \( d \)-dimensional measurable linear cocycle over \( \alpha \) is a measurable map

\[
\mathcal{A} : G \times X \to \mathrm{GL}(d, \mathbb{R})
\]

satisfying (almost surely) the cocycle conditions:

1. \( \mathcal{A}(e, x) = \text{Id} \);
2. for all \( g_1, g_2 \in G \)

\[
\mathcal{A}(g_1 g_2, x) = \mathcal{A}(g_1, \alpha(g_2)(x)) \mathcal{A}(g_2, x).
\] (1)

We say two cocycles \( \mathcal{A}, \mathcal{B} : G \times X \to \mathrm{GL}(d, \mathbb{R}) \) are (measurably) cohomologous if there is a measurable map \( \Phi : X \to \mathrm{GL}(d, \mathbb{R}) \) such that

\[
\mathcal{B}(g, x) = \Phi(\alpha(g)(x))^{-1} \mathcal{A}(g, x) \Phi(x).
\]

We say a cocycle \( \mathcal{A} : G \times X \to \mathrm{GL}(d, \mathbb{R}) \) is constant \( \mathcal{A}(g, x) \) is independent of \( x \) almost surely. Note that a constant cocycle then coincides with a representation \( \pi : G \to \mathrm{GL}(d, \mathbb{R}) \).

As a primary example, let \( \alpha : G \to \mathrm{Diff}^1_{\mu}(M) \) be an action of \( G \) by \( C^1 \) diffeomorphisms of a compact manifold \( M \) preserving some Borel probability \( \mu \). Although the tangent bundle \( TM \) may not be a trivial bundle, we may choose a Borel measurable identification
of vector-bundles $\Psi: TM \to M \times \mathbb{R}^d$ where $d = \dim(M)$. We fix such an identification $\Psi$ for the remainder.

We define $\mathcal{A}$ to be the derivative cocycle relative to this identification:

$$\mathcal{A}(g, x) = D_x \alpha(g)$$

where, we view $D_x \alpha(g)$ as an element of $\text{GL}(d, \mathbb{R})$ transferring the fiber $\{x\} \times \mathbb{R}^d$ to $\{\alpha(g)(x)\} \times \mathbb{R}^d$ via the measurable identification $\Psi$. To be precise if $\Psi: TM \to M \times \mathbb{R}^d$ is the measurable vector-bundle identification then

$$\mathcal{A}(g, x) := \Psi(\alpha(g)(x)) D_x \alpha(g) \Psi(x)^{-1}.$$ 

In this case, the cocycle relation (1) is simply the chain rule.

We have the following elementary claim.

**Claim 4.1.** Let $\alpha: G \to \text{Diff}_{\text{vol}}^1(M)$ be an action by volume preserving diffeomorphisms. Then the derivative cocycle $\mathcal{A}$ is cohomologous to and $\text{SL}(d, \mathbb{R})$-valued cocycle.

### 4.2. Cocycle Superrigidity

We formulate the statement of Zimmer’s cocycle superrigidity theorem.

**Theorem 4.2** (Cocycle superrigidity [Zim1]). For $n \geq 3$, let $G$ be either $G = \text{SL}(n, \mathbb{R})$ or let $G$ be a lattice in $\text{SL}(n, \mathbb{R})$. Let $\alpha: G \to \text{Aut}(X, \mu)$ be an action of $G$ by measurable $\mu$-preserving transformations of a standard probability space $(X, \mu)$. Let $A: G \times X \to \text{GL}(d, \mathbb{R})$ be a linear cocycle over the action $\alpha$.

Then there exist

1. a linear representation $\rho: \text{SL}(n, \mathbb{R}) \to \text{SL}(d, \mathbb{R})$;
2. a compact subgroup $K \subset \text{GL}(d, \mathbb{R})$ that commutes with the image of $\rho$;
3. and a measurable function $\Phi: X \to \text{GL}(d, \mathbb{R})$

such that for a.e. $x \in X$ and every $g \in G$

$$\mathcal{A}(g, x) = \Phi(\alpha(g)(x))^{-1} \rho(g) k \Phi(x)$$

(2)

for some $k \in K$.

In (2) one can verify that

$$(g, x) \mapsto \rho(g)^{-1} \Phi(\alpha(g)(x)) \mathcal{A}(g, x) \Phi(x)^{-1} \in K$$

is a cocycle. In particular, Theorem 4.2 can be reformulated as follows:

**Theorem 4.2’.** For $\alpha, G$, and $X$ as in Theorem 4.2, any linear cocycle $A: G \times X \to \text{GL}(d, \mathbb{R})$ over the action $\alpha$ is cohomologous to the product of constant cocycle $\rho: G \to \text{SL}(d, \mathbb{R})$ and a compact-valued cocycle $C: G \times X \to K \subset \text{GL}(d, \mathbb{R})$. Moreover, the images of $C$ and $\rho$ commute.

### 4.3. Superrigidity for Homomorphisms

Zimmer’s cocycle superrigidity theorem is an extension of Margulis’s superrigidity theorem for homomorphisms.

**Theorem 4.3** (Margulis superrigidity [Mar2]). For $n \geq 3$, let $\Gamma$ be a lattice in $\text{SL}(n, \mathbb{R})$. Given a representation $\rho: \Gamma \to \text{GL}(d, \mathbb{R})$ there are

1. a linear representation $\hat{\rho}: \text{SL}(n, \mathbb{R}) \to \text{SL}(d, \mathbb{R})$;
2. a compact subgroup $K \subset \text{GL}(d, \mathbb{R})$ that commutes with the image of $\hat{\rho}$ such that

$$\hat{\rho}(\gamma) \rho(\gamma)^{-1} \in K$$

for all $\gamma \in \Gamma$.  


That is, \( \rho = \hat{\rho} \cdot c \) is the product of the restriction of a representation \( \hat{\rho} : \text{SL}(n, \mathbb{R}) \to \text{SL}(d, \mathbb{R}) \) to \( \Gamma \) and a compact-valued representation \( c : \Gamma \to K \). Moreover the image of \( \hat{\rho} \) and \( c \) commute.

In the case that \( \Gamma \) is non-uniform, one can show that all compact-valued representations \( c : \hat{\rho} \to K \) have finite image. In the case that \( \Gamma \) is cocompact, there exists compact-valued representations \( c : \Gamma \to \text{SU}(n) \) with infinite image. However, the following consequence of Margulis’s superrigidity and arithmeticity theorems [Mar2]—which characterizes homomorphisms from lattices into compact Lie groups—shows that representations into \( \text{SU}(n) \) are more-or-less the only such examples.

**Theorem 4.4.** For \( n \geq 3 \), let \( \Gamma \subset \text{SL}(n, \mathbb{R}) \) be a lattice. Let \( K \) be a compact Lie group and \( \pi : \Gamma \to K \) a homomorphism.

1. If \( \Gamma \) is non-uniform then \( \pi(\Gamma) \) is finite.
2. If \( \Gamma \) is cocompact and \( \pi(\Gamma) \) is infinite then there is a closed subgroup \( K' \subset K \) with
   \[
   \pi(\Gamma) < K' \subset K
   \]
   and the Lie algebra of \( K' \) is of the form \( \text{Lie}(K') = \text{su}(n) \times \cdots \times \text{su}(n) \).

The appearance of \( \text{su}(n) \) in (2) of Theorem 4.4 is due to the fact that \( \text{su}(n) \) is the compact real form of \( \text{sl}(n) \).

### 4.4. Early evidence for Conjecture 3.3.

Note that if \( d < n \) then there are no representations \( \hat{\rho} : \text{SL}(n, \mathbb{R}) \to \text{SL}(d, \mathbb{R}) \) and there is no copy of \( \text{su}(n) \) in \( \text{sl}(d, \mathbb{R}) \). In particular, we immediately obtain as corollaries of Theorems 4.3 and 4.4 the following.

**Corollary 4.5.** For \( n \geq 3 \), let \( \Gamma \) be a lattice in \( G = \text{SL}(n, \mathbb{R}) \). Then, for \( d < n \) the image of any representation \( \rho : \Gamma \to \text{GL}(d, \mathbb{R}) \) is finite.

Conjecture 3.3 can be seen a non-linear extension of this corollary.

Now for \( n \geq 3 \), let \( \Gamma \) be a lattice in \( G = \text{SL}(n, \mathbb{R}) \) and consider \( \alpha : \Gamma \to \text{Diff}^1_{\mu}(M) \) where \( M \) is a compact manifold of dimension at most \( n - 1 \) and \( \mu \) is an arbitrary Borel probability measure on \( M \) preserved by \( \alpha \). The derivative cocycle over \( \alpha \) is then \( \text{GL}(n - 1, \mathbb{R}) \)-valued. Since there are no representation \( \rho : \text{SL}(n, \mathbb{R}) \to \text{SL}(n - 1, \mathbb{R}) \) Theorem 4.2 implies that the derivative cocycle is cohomologous to a compact-valued cocycle. In particular, we have the following:

**Corollary 4.6.** For \( \Gamma, M, \mu \) and \( \alpha : \Gamma \to \text{Diff}^1_{\mu}(M) \) as above

1. \( \alpha \) preserves a \( \mu \)-measurable Riemannian metric; i.e. there is a \( \mu \)-measurable, \( \alpha \)-invariant, symmetric positive-definite two-form on \( TM \).
2. For any \( \varepsilon > 0 \) and \( \gamma \in \Gamma \), the set of \( x \in \mathbb{X} \) such that
   \[
   \|D_x \alpha(\gamma^n)\| \geq \varepsilon^n
   \]
   has zero measure.

(1) follows from pulling back a \( K \)-invariant inner product on \( \mathbb{R}^d \) to \( TM \). (2) follows from Poincaré recurrence to sets where the function \( \Phi \) in Theorem 4.2 has bounded norm and conorm. Note that from (2), all Lyapunov exponents for individual elements of the action must vanish.

From Corollary 4.6, for \( n \geq 3 \) and \( \Gamma \) a lattice in \( G = \text{SL}(n, \mathbb{R}) \) we have that any action \( \alpha : \Gamma \to \text{Diff}^1_{\text{mea}}(M) \) preserves a measurable Riemannian metric \( g \) when \( M \) is a compact manifold of dimension at most \( n - 1 \). Suppose one could show that \( g \) was continuous or
As we discuss in Step 3 of Section 11 below, this combined with Theorem 4.4 implies the image $\alpha(\Gamma)$ is finite.

Thus Conjecture 3.3 follows if one can promote the measurable invariant metric $g$ guaranteed by Theorem 4.2 is continuous. In addition to solutions to Conjecture 3.3 in very low dimensions or with additional regularity, this is very strong evidence that the general conjecture given in Conjecture 3.3 is plausible.

Part 2. Smooth ergodic theory of $\mathbb{Z}^d$-actions and Proof of Theorem 3.8

We review in the first three sections of this part a number of classical results in smooth ergodic theory and outline how they extend to the case of actions of higher-rank abelian groups.

5. Lyapunov exponents for non-uniformly hyperbolic abelian actions

5.1. Lyapunov exponents a single diffeomorphisms. Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism of a compact manifold $M$. Let $\mu$ be an ergodic, $f$-invariant, Borel probability measure.

**Theorem 5.1.** There are

1. a measurable set $\Lambda$ with $\mu(\Lambda) = 1$;
2. numbers $\lambda^1 > \lambda^2 > \cdots > \lambda^p$; and
3. a $\mu$-measurable, $Df$-invariant splitting $T_x M = \bigoplus_{i=1}^p E^i(x)$ defined for $x \in \Lambda$ such that for every $x \in \Lambda$
   
   (a) for every $v \in E^i(x) \setminus \{0\}$
   
   $$\lim_{n \to \pm\infty} \frac{1}{n} \log \|D_x f^n(v)\| = \lambda^i;$$

   (b) if $Jf$ denotes the Jacobian determinant of $f$ then
   
   $$\lim_{n \to \pm\infty} \frac{1}{n} \log |Jf^n| = \sum_{i=1}^p \lambda^i;$$

   (c) for every $i \neq j$ we have
   
   $$\lim_{n \to \pm\infty} \frac{1}{n} \log \left( \sin \angle \left( E^i(f^n(x)), E^j(f^n(x)) \right) \right) = 0.$$

The numbers $\lambda^i$ are called **Lyapunov exponents** of $f$ with respect to $\mu$ and the subspaces $E^i(x)$ are called the **Oseledec’s subspaces**. We write $m^i$ for the almost-surely constant value of $\dim E^i(x)$, called the **multiplicity** of $\lambda^i$.

We write

$$E^u(x) = \bigoplus_{\lambda_i > 0} E^i(x), \quad E^c(x) = \bigoplus_{\lambda_i = 0} E^i(x), \quad E^s(x) = \bigoplus_{\lambda_i < 0} E^i(x)$$

for the **unstable**, **neutral**, and **stable** Oseledec’s subspaces.

Given any $f$-invariant measure $\mu$ on $M$ (which may be non-ergodic) the **average top fiberwise Lyapunov exponent** of $f$ with respect to $\mu$ is

$$\lambda_{\text{top}}(f, \mu) = \inf_{n \to \pm\infty} \frac{1}{n} \int \log \|D_x f^n\| \, d\mu(x). \quad (3)$$

Note that if $\{\mu^e_x\}$ is the partition of $\mu$ into $f$-ergodic components and if $\lambda^{1}_x > \lambda^{2}_x > \cdots > \lambda^{p(x)}_x$ are the Lyapunov exponents of $f$ with respect to the ergodic invariant measures $\mu^e_x$ then

$$\lambda_{\text{top}}(f, \mu) = \int \lambda^1_x \, d\mu(x).$$
5.2. **Exponents and exponential growth of derivatives.** Let $M$ be a compact manifold equipped with a fixed Riemannian metric. Let $f: M \to M$ be a $C^1$ diffeomorphism. We say $f: M \to M$ has *uniform subexponential growth of derivatives* if for all $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that
\[
\|Df^n\| := \sup_{x \in M} \|D_x f^n\| < C_\varepsilon e^{\varepsilon |n|} \quad \text{for all } n \in \mathbb{Z}.
\]

Note that we allow (and in applications expect) $C_\varepsilon \to \infty$ as $\varepsilon \to 0$.

**Proposition 5.2.** A diffeomorphism $f: M \to M$ has uniform subexponential growth of derivatives if and only if all Lyapunov exponents of $f$ with respect to any $f$-invariant measure $\mu$ are zero for any $f$-invariant measures $\mu$.

That is, $f: M \to M$ has uniform subexponential growth of derivatives if and only if
\[
\lambda_{\text{top}}(f, \mu) = \lambda_{\text{top}}(f^{-1}, \mu) = 0
\]
for all $f$-invariant measures $\mu$.

**Proof.** Suppose that $f: M \to M$ fails to have uniform subexponential growth of derivatives. Then there is an $\varepsilon > 0$ and a sequence of iterates $n_j \in \mathbb{Z}$, base points $x_j \in M$, and unit vectors $v_j \in T_{x_j}M$ with $|n_j| \to \infty$ such that
\[
\|D_{x_j} f^{n_j} v_j\| \geq e^{\varepsilon |n_j|}.
\]

Replacing $f$ with $f^{-1}$, we may assume without loss of generality that $n_j > 0$ for all $j$.

Let $U \subset TM$ be the unit-sphere bundle. We represent an element of $U$ by a pair $(x, v)$ where $v \in T_x M$ with $\|v\| = 1$. Note that $U$ is compact. Note also that $Df: TM \to TM$ induces a map $Uf: U \to U$ given by renormalizing the derivative:
\[
Uf(x, v) = \frac{D_x f(v)}{\|D_x f(v)\|}.
\]

Given $(x, v) \in U$, let
\[
\Phi(x, v) = \log \|D_x f(v)\|.
\]

By the chain rule, we have
\[
\log \|D_x f^n(v)\| = \sum_{j=0}^{n-1} \Phi(Uf^j(x, v)).
\]

For each $j$, let $\nu^j$ be the measure on $U$ given by
\[
\nu^j = \frac{1}{n_j} \sum_{k=0}^{n_j-1} \delta_{Uf^k(x_j, v_j)}.
\]

Note that (4) implies that $\int \Phi \, d\nu^j \geq \varepsilon$ for each $j$. Consider any weak-$*$ subsequential limit $\nu$ of $\{\nu^j\}$. We claim

**Claim 5.3.**

(a) $\nu$ is $Uf$-invariant;

(b) $\int \Phi \, d\nu \geq \varepsilon$.

(a) follows as in the proof of the Krylov-Bogolyubov theorem. (b) follows from continuity of $\Phi$ and weak-$*$ convergence. From (b), we may replace $\nu$ with an ergodic component of $\nu'$ such that $\int \Phi \, d\nu' \geq \varepsilon$.

Take $\mu$ to be the measure on $M$ given by push-forward of $\nu'$ under the natural projection $U \to M$. Let $\{\nu'_x\}$ denote a family of conditional measures of $\nu'$ for the partition of $U$ into fibers over $M$. By the pointwise ergodic theorem, for $\mu$-a.e. $x \in M$ and $\nu'_x$-a.e. $v \in U(x)$
we have
\[
\lim_{n \to \infty} \frac{1}{n} \log \|D_x f^n(v)\| = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \Phi(U f^j(x, v)) = \int \Phi \, d\nu' \geq \epsilon.
\]

It follows from (14) that the largest Lyapunov exponent of \( f \) with respect to \( \mu \) is at least \( \epsilon > 0 \). Taking the contrapositive, it follows that if all Lyapunov exponents of \( f \) with respect any \( f \)-invariant measure are zero then \( f: M \to M \) has uniform subexponential growth of derivatives.

The converse direction is left to the reader. \( \square \)

5.3. Lyapunov exponents for nonuniformly hyperbolic \( \mathbb{Z}^d \)-actions. For actions of higher-rank abelian groups there is an analogue of Theorem 5.1. In the rank-1 setting, Lyapunov exponents are numbers. In the higher-rank setting, they are replaced with linear functionals. We state our results for the case of discrete actions (i.e. actions of \( \mathbb{Z}^d \)) but similar results hold for actions of more general free abelian groups of finite rank such as \( \mathbb{Z}^k \times \mathbb{R}^\ell \).

5.3.1. Lyapunov exponents. Let \( M \) be a compact manifold, let \( \alpha: \mathbb{Z}^d \to \text{Diff}^{1+\alpha}(M) \) be a smooth \( \mathbb{Z}^d \)-action, and let \( \mu \) be an ergodic, \( \alpha \)-invariant measure.

**Theorem 5.4** (Higher-rank Oseledec’s theorem). There are

1. a measurable set \( \Lambda \) with \( \mu(\Lambda) = 1 \);
2. linear functionals \( \lambda^1, \lambda^2, \ldots, \lambda^p: \mathbb{R}^d \to \mathbb{R} \); and
3. a \( \mu \)-measurable, \( \text{Diff} \)-invariant splitting \( T_x M = \bigoplus_{i=1}^p E^i(x) \) defined for \( x \in \Lambda \)

such that for every \( x \in \Lambda \)

(a) for every \( v \in E^i(x) \setminus \{0\} \)
\[
\lim_{|n| \to \infty} \frac{\log \|D_x \alpha(n)(v)\| - \lambda^i(n)}{|n|} = 0;
\]

(b) if \( Jf \) denotes the Jacobian determinant of \( f \) then
\[
\lim_{|n| \to \infty} \frac{\log |J\alpha(n)| - \sum_{i=1}^p \lambda^i(n)}{|n|} = 0;
\]

(c) for every \( i \neq j \) \( \lim_{n \to \infty} \frac{1}{|n|} \log \left( \sin \angle \left( E^i(\alpha(n)(x)), E^j(\alpha(n)(x)) \right) \right) = 0. \)

Note that (a) implies convergence along rays: for any \( n \in \mathbb{Z}^d \) and \( v \in E^i(x) \setminus \{0\} \)
\[
\lim_{k \to \infty} \frac{1}{k} \log \|D_x \alpha(kn)(v)\| = \lambda^i(n). \quad (5)
\]
The convergence in (a) is taken along any sequence \( n \to \infty \); this is stronger than (5) and is typically needed in applications.

To prove Theorem 5.4 from Theorem 5.1 one can apply Theorem 5.1 to each of the (countably many) rays in \( \mathbb{Z}^n \) to obtain convergence of the limits in (5) and the splittings in Theorem 5.4. That the functions \( \lambda^i \) are linear follows from the cocycle property (i.e. chain rule) of the derivative. Using that the derivative cocycle is bounded (or more generally is \( L^{d,1} \)) one can show the convergence along spheres in (a) of Theorem 5.4

5.3.2. Coarse Lyapunov exponents and coarse Lyapunov subspaces. Recall that for a single diffeomorphism, we groups exponents by sign and the corresponding subspaces into stable, neutral, and unstable subspaces. What should play the role of stable and unstable subspaces for \( \mathbb{Z}^d \)-actions?

Given Lyapunov exponents \( \lambda^1, \lambda^2, \ldots, \lambda^p: \mathbb{R}^d \to \mathbb{R} \) we say \( \lambda^i \) and \( \lambda^j \) are **coarsely equivalent** if there is a \( c > 0 \) with
\[
\lambda^i = c \lambda^j.
\]
Note that this is an equivalence relation on the linear functionals $\lambda^1, \lambda^2, \ldots, \lambda^p : \mathbb{R}^d \to \mathbb{R}$. The coarse equivalence classes are called coarse Lyapunov exponents. For $\mathbb{Z}$-actions generated by a diffeomorphism $f : M \to M$, the coarse Lyapunov exponents are simply the collections of positive, zero, and negative Lyapunov exponents.

Let $\chi = \{ \lambda^1 \}$ be a coarse Lyapunov exponent. Write

$$E^\chi(x) = \oplus_{\lambda, e^\chi} E^\lambda(x)$$

for the corresponding coarse Lyapunov subspace. Note that for $n \in \mathbb{Z}^d$, while the size of $\chi(n)$ is not well defined, the sign of $\chi$ is well defined.

Given the triple $(M, \mu, \alpha)$, a Weyl chamber wall is a hyperplane of the form $\ker \chi$ for some coarse Lyapunov exponent $\chi$ of $\alpha$ with respect to $\mu$. A Weyl chamber is a connected component $W$ of $\mathbb{R}^d \setminus \bigcup \ker \chi$ where the union is taken over all non-trivial coarse Lyapunov exponents. In other words, a Weyl chamber is a maximal collection $W$ of elements in $\mathbb{R}^d$ such that for each given $\chi$, $\chi(t)$ has the same sign for all elements $t \in W$.

6. Unstable and coarse Lyapunov Manifolds

6.1. Unstable manifolds for a single diffeomorphism. Through $\mu$-almost every point $x$ the set

$$W^u(x) := \left\{ y : \limsup_{n \to -\infty} d(f^n(x), f^n(y)) < 0 \right\}$$

is a connected $C^{1+\alpha}$ injectively immersed manifold with $T_xW^u(x) = E^u(x)$ (see [Pes1]) called the (global) unstable manifold of $f$ through $x$. The collection of all $W^u(x)$ forms a partition of $M$ though, in general, this partition does not have the structure of a nice foliation. However, restricted to sets of large measure the partition into local unstable manifolds has a continuous lamination structure.

6.2. Coarse Lyapunov manifolds for $\mathbb{Z}^d$-actions. Analogous to the existence and properties of unstable Pesin manifolds for non-uniformly hyperbolic diffeomorphisms we have the following.

**Proposition 6.1.** For almost every $x \in \Lambda$ and for every coarse Lyapunov exponent $\chi$ there is a connected, $C^{1+\alpha}$, injectively immersed manifold $W^\chi(x)$ satisfying the following:

1. $T_xW^\chi(x) = E^\chi(x)$;
2. $\alpha(n)W^\chi(x) = W^\chi(\alpha(n)(x))$ for all $n \in \mathbb{Z}^d$;
3. $W^\chi(x)$ is the set of all $y \in M$ satisfying

$$\limsup_{k \to -\infty} \frac{1}{k} \log d(\alpha(kn)(y), \alpha(kn)(x)) < 0$$

for all $n \in \mathbb{Z}^d$ with $\chi(n) > 0$.

To see the existence of $W^\chi$, it suffices to take the common intersection $\bigcap_{n \in \mathbb{Z}^d} W^u_{\alpha(n)}(x)$, where the intersection is taken over all $n \in \mathbb{Z}^d$ with $\chi(n) > 0$.

6.3. Partitions subordinate to a foliation. For a $\mathbb{Z}^d$-action, we shall consider measurable partitions that characterize dynamically defined foliations such as $W^u$ and $W^\chi$.

**Definition 6.2.** A measurable partition $\eta$ of $(M, \mu)$ is subordinate to a foliation $\mathcal{F}$ with smooth leaves if for $\mu$-almost every $x$, $\eta(x)$ is a bounded neighborhood of $x$ in the smooth manifold $\mathcal{F}(x)$.

For a dynamically defined foliations, we can construct such partitions with additional dynamical properties.
Lemma 6.3. (following [Hu, Section 8]) If $\mathcal{F}$ is an $\alpha$-invariant measurable, $C^{1+\text{Hölder}}$ tame foliation\(^3\) Given finitely many $n_1, \ldots, n_l \in \mathbb{Z}^d$ such that $\mathcal{F}(x) \subset W^u_{\alpha(n_i)}(x)$, there exists a measurable partition $\xi^{\mathcal{F}}$ of $(M, \mu)$ such that:

1. $\xi^{\mathcal{F}}$ is subordinate to $\mathcal{F}$;
2. $\xi^{\mathcal{F}}$ is $\alpha(n_i)$-decreasing: $\alpha(n_i)\xi^{\mathcal{F}} \leq \xi^{\mathcal{F}}$;
3. $\xi^{\mathcal{F}}$ is $\alpha(n_i)$-generating: $\vee_{k=0}^{\infty} \alpha(-kn_i)\xi^{\mathcal{F}}$ is the point partition.

7. Metric entropy

Let $(X, \mu)$ be a standard probability space. Given a measurable partition $\xi$ of $(X, \mu)$, let $\{\mu^n_{\xi}\}$ denote a family of conditional measures with respect to $\xi$. That is,

1. each $\mu^n_{\xi}$ is a Borel probability on $X$;
2. $x \mapsto \mu^n_{\xi}(A)$ is measurable for any Borel $A \subset X$;
3. $\mu(A) = \int \mu^n_{\xi}(A)\,d\mu(x)$ for any Borel $A \subset X$.

Given measurable partitions $\eta, \xi$ of $(X, \mu)$, the mean conditional information of $\eta$ relative to $\xi$ is $I_{\mu}(\eta \mid \xi)(x) = -\log(\mu^n_{\xi}(\eta(x)))$ and the mean conditional entropy of $\eta$ relative to $\xi$ to be

$$H_{\mu}(\eta \mid \xi) = \int I_{\mu}(\eta \mid \xi)(x)\,d\mu(x).$$

The entropy of $\eta$ is $H_{\mu}(\eta) = H_{\mu}(\eta \mid \emptyset, X)$. Note that if $H_{\mu}(\eta) < \infty$ then $\eta$ is necessarily countable. Let $f : (X, \mu) \to (X, \mu)$ be an invertible, measurable, measure-preserving transformation. Let $\eta$ be an arbitrary measurable partition of $(X, \mu)$. We define

$$\eta^+ := \bigvee_{i=0}^{\infty} f^i\eta, \quad \eta^- := \bigvee_{i\in\mathbb{Z}} f^i\eta.$$

We define the entropy of $f$ given the partition $\eta$ to be

$$h_{\mu}(f, \eta) := H_{\mu}(\eta \mid f\eta^+) = H_{\mu}(\eta^+ \mid f\eta^+) = H_{\mu}(f^{-1}\eta^+ \mid \eta^+).$$

We define the $\mu$-entropy of $f$ to be $h_{\mu}(f) = \sup\{h_{\mu}(f, \eta)\}$ where the supremum is taken over all measurable partitions of $(X, \mu)$.

More generally, for a partition $\eta$, the $\mu$-entropy of $f$ subordinate to $\eta$ is

$$h_{\mu}(f\mid\eta) = \sup\{h_{\mu}(f, \xi) : \xi \geq \eta\}.$$

For an $f$-invariant foliation $\mathcal{F}$, the $\mu$-entropy of $f$ subordinate to $\mathcal{F}$ is

$$h_{\mu}(f\mid\mathcal{F}) = \sup\{h_{\mu}(f, \xi) : \xi \text{ subordinate to } \mathcal{F}\}.$$

A fundamental identity between entropies is that $h_{\mu}(f\mid\mathcal{F}) = h_{\mu}(f\mid\mathcal{F} \cup \mathcal{W}^u)$.

7.1. Entropy, exponents, and geometry of conditional measures. We recall the following definition. Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism and let $\mu$ be an ergodic, $f$-invariant measure.

Definition 7.1. We say $\mu$ is an SRB measure (or satisfies the SRB property) if, for any measurable partition $\xi$ of $(M, \mu)$ subordinate to the partition into unstable manifolds, for almost every $x$ the conditional measure $\mu^n_{\xi}$ is absolutely continuous with respect to Riemannian volume on $W^u(x)$.

We have the following summary of a number of important results.

\(^3\)C^{1+\text{Hölder}}\) tame means the submanifolds $\mathcal{F}(x)$ are $C^{1+\text{Hölder}}$ and the $C^{1+\text{Hölder}}$ constants degenerates slowly along orbits. See [BRH, Definition 4.3].
Theorem 7.2. Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism and let $\mu$ be an ergodic, $f$-invariant measure. Then

1. $h_{\mu}(f) \leq \sum_{\lambda_i > 0} m^i \lambda_i$;
2. if $\mu$ is absolutely continuous with respect to volume, then $h_{\mu}(f) = \sum_{\lambda_i > 0} m^i \lambda_i$;
3. if $\mu$ is SRB, then $h_{\mu}(f) = \sum_{\lambda_i > 0} m^i \lambda_i$.

Theorem 7.2(1), known as the Margulis–Ruelle inequality, is proven in [Rue]. Theorem 7.2(2), known as the Pesin entropy formula, is shown in [Pes2]. Theorem 7.2(3) was established by Ledrappier and Strelcyn in [LS]. In the next section, we will complete Theorem 7.2 with Ledrappier’s Theorem, Theorem 8.3, which provides a converse to Theorem 7.2(3).

For general measures invariant under a $C^2$-diffeomorphism Ledrappier and Young explain explicitly the defect from equality in Theorem 7.2(1). Let $\delta^i$ denote the (almost-surely constant value of the) pointwise dimension of $\mu$ along the $i$th unstable manifold. With $\delta^0 = 0$ let

$$\gamma^i = \delta^i - \delta^{i-1}. \quad (6)$$

The coefficients $\gamma^i$ reflect the geometry of the measure $\mu$ inside of the unstable manifolds.

Theorem 7.3. [LY2] Let $f : M \to M$ be a $C^2$ diffeomorphism and let $\mu$ be an ergodic, $f$-invariant measure. Then $h_{\mu}(f) = \sum_{\lambda_i > 0} \gamma^i \lambda_i$.

7.2. Product structure of entropy for higher-rank actions. Once nice feature of dynamics of higher-rank abelian groups is the product structures between conditional measures along different foliations. This property, called product lemma, was first introduced by Einsiedler and Katok in [EK1] in the homogeneous setting. The original product lemma can be explained as follows using our terminology so far:

For a $\mathbb{Z}^d$-action $\alpha$ on $(M, \mu)$, if two distinct coarse Lyapunov exponents $\chi, \theta$ both take positive values on $n \in \mathbb{Z}^d$, Moreover, for simplicity, suppose there is $m \in \mathbb{Z}^d$ such that $\chi(m) = 0$ but $\theta(m) > 0$. Suppose in addition that there is a measurable family of chart maps $\phi_x : W^\chi(x) \to \mathbb{R}^{\dim W^\chi}$, such that $\alpha(m)$ an isometry between $W^\chi(x)$ and $W^\chi(\alpha(m)x)$ for almost every $x$ after conjugating by $\phi_x$ on both sides. (For simplicity, one may assume the dynamics of $\alpha(m)$ is in fact isometric on $W^\chi$-leaves.) Then for generic points $x, x'$, if $x$ and $x'$ are in the same $W^\theta$-leaf, then $\mu^{W^\chi}_{\phi_x}$ and $\mu^{W^\chi}_{\phi_{x'}}$ coincide, after identifying both $W^\chi(x)$ and $W^\chi(x')$ with $\mathbb{R}^{\dim W^\chi}$ using $\phi_x$ and $\phi_{x'}$. If there exits a foliation $\mathcal{F}$ that is tangent to $E^\chi \oplus E^\theta$. Then this factor shows $\mu^{W^\chi}_{\phi_x}$ can be decomposed as a product between $\mu^{W^\chi}_{\phi_x}$ and $\mu^{W^\theta}_{\phi_x}$, almost for every $x$. This in particular implies that $h(\alpha(n)|\mathcal{F}) = h(\alpha(n)|W^\chi) + h(\alpha(n)|W^\theta)$.

If the isometric condition on $W^\chi$ is dropped, such exact product structures between conditional measures are no longer available. However, we can prove that they remain valid in the weaker sense of conditional entropies.

Theorem 7.4. If $\eta$ is an $\alpha$-invariant measurable partition of $(M, \mu)$, then for $n \in \mathbb{Z}^d$,

$$h_{\mu}(\alpha(n)|\eta) = \sum_{x : \chi(n) > 0} h_{\mu}(\alpha(n)|W^\chi \cup \eta).$$

Explanation of the proof. To better explain the proof, let us assume for simplicity $\mathcal{F} = \mathcal{W}^m_n$, for some $m$ and $\eta$ is the trivial partition. Moreover, assume that $E^m_n$ is the direct sum $E^{\lambda_1} \oplus E^{\lambda_2}$ of two distinct Lyapunov spaces. We can assume $\lambda_1(m) > \lambda_2(m) > 0$.\[\]
For $\alpha(m)$, the first and second unstable foliations are respectively $W^{\lambda_1}$ and $\mathcal{F}$. Denote by $\delta^1$ and $\delta^2$ the (almost surely constant) pointwise dimensions of $\mu$ along these foliations. According to (6), $\gamma^1 = \delta^1$ and $\gamma^2 = \delta^2 - \delta^1$.

One can find a different element $m_*$ from the same Weyl chamber $W$ containing $n$ such that $\lambda_2(m_*) > \lambda_1(m_*) > 0$. The first and second unstable manifolds for $\alpha(m_*)$ are respectively $W^{\lambda_2}$ and $\mathcal{F}$. Let $\delta^1_*$ and $\delta^2_*$ be the corresponding pointwise dimensions, and $\gamma^1_* = \delta^1_*$ and $\gamma^2_* = \delta^2_* - \delta^1_*$. It should be noted that $\delta^2_* = \delta^2$.

By the Ledrappier-Young formula [LY2], the entropies of $\alpha(m)$ and $\alpha(m_*)$ are respectively $\gamma^1 \lambda_1(m) + \gamma^2 \lambda_2(m)$ and $\gamma^1_* \lambda_2(m_*) + \gamma^2_* \lambda_1(m_*)$. On the other hand, by a result of Hu [Hu], the entropy is a linear function in the same Weyl chamber. Because $\lambda_1$, $\lambda_2$ are linearly independent, and $n$, $n_*$ are in the same Weyl chamber and each of them can be perturbed in a neighborhood, in order to have linearity we must have $\gamma^1 = \gamma^2_*$ and $\gamma^1_* = \gamma^2$.

It follows that $\delta^2 = \gamma^1 + \gamma^2 = \gamma^1 + \gamma^1_*$. Equivalently, the dimension of the conditional measure of $\mu$ along $\mathcal{F}$ is the same of the dimensions of its conditional measures along $W^{\lambda_1}$ and $W^{\lambda_2}$.

Finally, for any $n$ in the same Weyl chamber $W$,
\[ h_\mu(\alpha(n)|\mathcal{F}) = \gamma^1 \lambda_1(n) + \gamma^2 \lambda_2(n) = \gamma^1 \lambda_1(n) + \gamma^1_* \lambda_2(n) = h_\mu(\alpha(n)|W^{\lambda_1}) + h_\mu(\alpha(n)|W^{\lambda_2}). \]

This is the simple special case of the Theorem that we are dealing with.

7.3. **Coarse-Lyapunov Abramov-Rokhlin formula.** One consequence of Theorem 7.4 is a version of Abramov-Rokhlin formula along coarse Lyapunov foliations. We assume that the $\mathbb{Z}^d$-action $(M, \mu)$ has a measurable factor $\tilde{\alpha} : \mathbb{Z}^d \curvearrowright (N, \nu)$ which itself is a smooth action.

A key fact is that:

**Lemma 7.5.** Every non-trivial coarse Lyapunov exponent $\chi$ of $(N, \nu)$ which satisfies that $h_\nu(\tilde{\alpha}(n)|\tilde{W}^x) > 0$ for at least one $n \in \mathbb{Z}^d$ must also be a coarse Lyapunov exponent of $(M, \mu)$.

Here $\tilde{W}^x$ denotes the corresponding coarse Lyapunov foliation on $N$. This requires some efforts to prove in the general case, but automatically hold in the settings of suspension actions over homogeneous spaces, which we will need later.

**Proposition 7.6** (Coarse Abramov-Rohlin formula, [BRHW3, Theorem 13.6]). For all $n \in \mathbb{Z}^d$ and $\chi$ as in the lemma above,
\[ h_\mu(\alpha(n)|W^x) = h_\mu(\alpha(n)|W^x \vee \mathcal{A}_N) + h_\nu(\tilde{\alpha}(n)|\tilde{W}^x), \]
where $\mathcal{A}_N$ denotes the point partition of the factor space $N$ realized as a partition of $M$.

The special case of Proposition that we are the most interested in, namely the projection is that between the suspension action of a $\Gamma$-action and the diagonal flow on $G/\Gamma$, will be explained as Theorem 10.3.

8. **Entropy, invariance, and the SRB property**

Absent zero Lyapunov exponents, in dissipative dynamical systems SRB measures provide examples of physical measures, that is measures which capture the statistical properties of the forward orbit of a Lebesgue positive measure set of points. A key problem in applications and for specific examples is to establish the existence of physical and SRB measures. We pose a related question:
**Question 8.1.** Given a diffeomorphism \( f : M \to M \) and an \( f \)-invariant measure \( \mu \), how do you verify that \( \mu \) is an SRB measure?

Seemingly unrelated, consider a group \( G \) acting smoothly on a manifold \( M \). We pose the following:

**Question 8.2.** Given a Borel probability measure \( \mu \) on \( M \) and a subgroup \( H \subset G \), how do you verify that \( \mu \) is \( H \)-invariant?

### 8.1. Ledrappier’s Theorem.
We outline one approach that solves both Question 8.1 and 8.2 in a number of settings. Given a diffeomorphism \( f : M \to M \) of a compact manifold and an \( f \)-invariant measure \( \mu \) one defines the metric entropy of \( f \) **conditioned on unstable manifolds**, denoted by \( h^u_{\mu}(f) \), as the supremum of \( h_{\mu}(f, \xi) \) where the supremum is taken over all measurable partitions \( \xi \) subordinate to unstable manifolds. The principle result (Corollary 5.3) of [LY1] shows that for \( C^2 \) diffeomorphisms\(^4\) we have equality of the metric entropy of \( f \) and the metric entropy of \( f \) conditioned on unstable manifolds:

\[
h_{\mu}(f) = h^u_{\mu}(f).
\]

**Theorem 8.3** (Ledrappier’s Theorem [Led1]). Let \( f \) be a \( C^{1+\alpha} \) diffeomorphism and let \( \mu \) be an ergodic, \( f \)-invariant, Borel probability measure. Then \( f \) is SRB if and only if

\[
h^u_{\mu}(f) = \sum_{\lambda_i > 0} m_i \lambda_i.
\]

In the proof of Theorem 8.3, Ledrappier actually proves something much stronger than the SRB property: if \( h^u_{\mu}(f) = \sum_{\lambda_i > 0} m_i \lambda_i \) then the conditional measures \( \mu^u_x \) of \( \mu \) along unstable manifolds are equivalent to the Riemannian volume with a Hölder continuous density. That is, if \( m^u_x \) the Riemannian volume along \( W^u(x) \) then for \( a.e. \ x \) there is a Hölder continuous (with respect to the internal metric topology of \( W^u(x) \)) function \( \rho : W^u(x) \to (0, \infty) \) with

\[
\mu^u_x = \rho \cdot m^u_x.
\]

Moreover, Ledrappier computes the density explicitly.

We make use of the explicit formula for the density \( \rho \) in the following setup. Consider a Lie group \( G \) and a smooth, locally free, action of \( G \) on a compact manifold \( M \). We denote the action by \( g \cdot x \) for \( g \in G \) and \( x \in M \). Consider a Lie subgroup \( H \subset G \) and \( s \in G \) that normalizes \( H \). Let \( f : M \to M \) be the diffeomorphism given by \( s \); that is \( f(x) = s \cdot x \). Let \( \mu \) be an ergodic, \( f \)-invariant Borel probability measure and suppose that the orbit \( H \cdot x \) is contained in the unstable manifold \( W^u(x) \) for \( \mu \)-almost every \( x \).

Note that since \( s \) normalizes \( H \), the partition of \( M \) into \( H \)-orbits is preserved by \( f \); in particular, the partition into \( H \)-orbits is a subfoliation of the partition into unstable manifolds. Given a Borel probability measure \( \mu \) on \( M \) and a measurable partition \( \xi \) subordinate to the partition into \( H \)-orbits we can define conditional measures \( \mu^u_x \) of \( \mu \). Given \( x \in M \) (using that the action is locally free) we can push forward the left-Haar measure on \( H \) onto the orbit \( H \cdot x \) via the parametrization \( H \cdot x = \{ h \cdot x : h \in H \} \).

**Lemma 8.4.** \( \mu \) is \( H \)-invariant if and only if for any measurable partition \( \xi \) subordinate to the partition into \( H \)-orbits and \( \mu \)-a.e. \( x \) the conditional measure \( \mu^u_x \) coincides up to normalization with the restriction of the left-Haar measure on \( H \cdot x \) to \( \xi(x) \).

---

\(^4\)For \( C^{1+\alpha} \)-diffeomorphism without zero Lyapunov exponents this equality was shown in Ledrappier’s article [Led1]; for the general case of \( C^{1+\alpha} \)-diffeomorphisms see [Bro].
Similarly to how we define the metric entropy of \( f \) conditioned on unstable manifolds we can define \( h_\mu(f \mid H) \), the metric entropy of \( f \) conditioned on \( H \)-orbits, as
\[
\sup h_\mu(f, \xi)
\]
where the supremum is taken over all measurable partitions \( \xi \) subordinate to \( H \)-orbits. Let \( \lambda^i, E^i(x) \), and \( m^i \) be as in Section 5.1 for the dynamics of \( f \) and the measure \( \mu \). We define the multiplicity of \( \lambda^i \) relative to \( H \) to be (the almost surely constant value of)
\[
m^{i,H} = \dim(E^i(x) \cap T_x(H \cdot x)).
\]
Generalizing Theorem 7.2(1) we have
\[
h_\mu(f \mid H) \leq \sum_{\lambda^i > 0} \lambda^i m^{i,H} \tag{8}
\]
From the proof of Theorem 8.3, (in particular the explicit formula for the density function \( \rho \)) we have the following.

**Theorem 8.5.** With the above setup, the following are equivalent:

1. \( h_\mu(f \mid H) = \sum_{\lambda^i > 0} \lambda^i m^{i,H} \);
2. for any measurable partition \( \xi \) subordinate to the partition into \( H \)-orbits and almost every \( x \), \( \mu_\xi \) is absolutely continuous with respect to the Riemannian volume on the \( H \)-orbit \( H \cdot x \);
3. \( \mu \) is \( H \)-invariant.

A possible critique of Theorem 8.3 is that in examples it seems nearly impossible to verify equality in (7) without a priori knowledge that the measure is SRB. However, in a number of settings of group actions on manifolds, it turns out one can in fact verify equality in (7) (or typically, equality in Theorem 8.5(1)) and thus derive the SRB property and gain additional invariance of the measure only from entropy considerations. This is the key idea in these notes, the papers [BRHW2, BFH], and also appears as a main tool in [EM, EK2, EKL].

**Remark 8.6.** The statement and proof of Theorem 8.3, especially the reformulation in Theorem 8.5, is very similar the invariance principle for fiberwise disintegrations of measures invariant under skew products. The earliest version of this invariance principle is due to Ledrappier [Led2] for projectivized linear cocycles. Avila-Viana extended this to cocycles taking values in the group of \( C^1 \) diffeomorphisms in [AV]. See Proposition 10.1 for a related invariance principle in the setting of actions of lattices on manifolds.

### 9. Structure theory of \( SL(n, \mathbb{R}) \) and Cartan flows on \( SL(n, \mathbb{R})/\Gamma \)

Let \( G = SL(n, \mathbb{R}) \) and let \( \Gamma \subset G \) be a lattice. Let \( G = KAN \) be the Iwasawa decomposition\(^5\) where
\[
K = SO(n, \mathbb{R}), \quad A = \{ \text{diag}(e^{t_1}, e^{t_2}, \ldots, e^{t_n}) : t_1 + \ldots + t_n = 1 \},
\]
and \( N \) is the group of strictly upper triangular matrices with 1s on the diagonal.

We will be interested in certain subgroups of \( G \) and how they capture dynamical information of the action of the cartan subgroup \( A \) on the homogeneous space \( G/\Gamma \).

---

\(^5\)relative to the Cartan involution \( g \mapsto (g^*)^{-1} \)
9.1. **Roots and root subgroups.** We consider the following linear functionals
\[ \beta^{i,j} : A \to \mathbb{R} \]
given as follows: for \( i \neq j \),
\[ \beta^{i,j}(\text{diag}(e^{t_1}, e^{t_2}, \ldots, e^{t_n})) = t_i - t_j. \]
The linear functionals \( \beta^{i,j} \) are the roots of \( G \).
Associated to each root \( \beta^{i,j} \) is a 1-parameter subgroup \( U^{i,j} \subset G \). For instance, in \( G = \text{SL}(3, \mathbb{R}) \) we have the following 1-parameter flows
\[
\begin{align*}
    u^{1,2}(t) &= \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
    u^{1,3}(t) &= \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
    u^{2,3}(t) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \\
    u^{2,1}(t) &= \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
    u^{3,1}(t) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t & 0 & 1 \end{pmatrix}, \\
    u^{3,2}(t) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix}.
\end{align*}
\]
We let \( U^{i,j} \) denote the associated 1-parameter unipotent subgroups of \( G \):
\[ U^{i,j} := \{ u^{i,j}(t) : t \in \mathbb{R} \}. \tag{9} \]
A fact that we will use in the sequel is that the subgroups \( U^{i,j} \) generate all of \( G \).

9.2. **Cartan flows.** For concreteness, consider \( G = \text{SL}(3, \mathbb{R}) \) and let \( \Gamma = \text{SL}(3, \mathbb{Z}) \). Let \( X \) denote the coset space \( X = G/\Gamma \). This is an 8 dimensional manifold (which is noncompact for \( \Gamma = \text{SL}(n, \mathbb{Z}) \)) \( G \) acts on \( X \) on the left: given \( g \in G \) and \( x = g\Gamma \in X \) we have
\[ g \cdot x = g\Gamma \in X. \]

The Cartan subgroup \( A \subset G \) is the subgroup of diagonal matrices with positive entries
\[ A := \left\{ \begin{pmatrix} e^{t_1} & 0 & 0 \\ 0 & e^{t_2} & 0 \\ 0 & 0 & e^{t_3} \end{pmatrix} : t_1 + t_2 + t_3 = 0 \right\}. \]
The group \( A \) is isomorphic to \( \mathbb{R}^2 \), for instance, via the embedding
\[ (s, t) \mapsto \text{diag}(e^s, e^t, e^{-s-t}) = \begin{pmatrix} e^s & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-s-t} \end{pmatrix}. \]

We will consider below the action \( \alpha : A \times X \to X \) of \( A \) on \( X \) given by
\[ \alpha(s)(x) = sx. \]

For the \( A \)-action on \( X \) and any \( A \)-invariant measures we have 6 non-zero\(^6\) Lyapunov exponents given by the roots
\[ \beta^{i,j} : A \to \mathbb{R}. \]
Note that no two roots \( \beta^{i,j} \) are positively proportional and hence are their own coarse equivalence class. For \( x \in X \) let \( W^{i,j}(x) \) be the orbit of \( x \) under the 1-parameter group \( U^{i,j} \):
\[ W^{i,j}(x) = \{ u^{i,j}(t)x : t \in \mathbb{R} \}. \]
We claim \( W^{i,j}(x) \) is the coarse Lyapunov manifold corresponding to \( \beta^{i,j} \). Indeed, if \( x \in X \) and \( x' = u^{i,j}(v)x \in W^{i,j}(x) \) then a linear algebra computation shows that for
\[ ^6\text{note there is a zero Lyapunov exponent with multiplicity two corresponding the } A\text{-orbits} \]
s = \text{diag}(e^{t_2}, e^{t_2}, e^{t_3}) \in A$, if

\[
\alpha(s)(x') = \text{diag}(e^{t_2}, e^{t_2}, e^{t_3})x'
\]

\[
= \text{diag}(e^{t_2}, e^{t_2}, e^{t_3})u^{i,j}(v)x
\]

\[
= u^{i,j}(v')\text{diag}(e^{t_2}, e^{t_2}, e^{t_3})x
\]

\[
= u^{i,j}(v')\alpha(s)(x)
\]

then \( v' = e^{\beta i,j}(\text{diag}(e^{t_2}, e^{t_2}, e^{t_3}))v \). That is, \( \alpha(s) \) dilates distances in \( W^{i,j} (x) \) by exactly \( e^{\beta i,j}(s) \). In particular, the tangent spaces to each \( W^{i,j}(x) \) as well as the tangent space to the orbit \( A \cdot x \) gives the \( A \)-invariant splitting guaranteed by Theorem 5.4.

10. Invariance Principle and Proof of Theorem 3.8

10.1. Suspension action. Fix \( G = \text{SL}(n, \mathbb{R}) \) and let \( \Gamma \subset G \) be a lattice. We may take \( \Gamma = \text{SL}(n, \mathbb{Z}) \).

Let \( M \) be a compact manifold and let \( \alpha : \Gamma \to \text{Diff}(M) \) be an action. On the product \( G \times M \) consider the right \( \Gamma \)-action

\[(g, x) \cdot \gamma = (g\gamma, \alpha(\gamma^{-1})(x))\]

and the left \( G \)-action

\[a \cdot (g, x) = (ag, x)\]

Define the quotient manifold \( M^\alpha := (G \times M)/\Gamma \). As the \( G \)-action on \( G \times M \) commutes with the \( \Gamma \)-action, we have an induced left \( G \)-action on \( M^\alpha \). For \( g \in G \) and \( x \in M^\alpha \) we denote this action by \( g \cdot x \) and denote the derivative of the diffeomorphism \( x \mapsto g \cdot x \) by \( D_xg : T_xM^\alpha \to T_gxM^\alpha \).

We write \( \pi : M^\alpha \to \text{SL}(n, \mathbb{R})/\Gamma \) for the natural projection map. Note that \( M^\alpha \) has the structure of a fiber-bundle over \( \text{SL}(n, \mathbb{R})/\Gamma \) induced by the map \( \pi \). The \( G \)-action permutes the \( M \)-fibers of \( M^\alpha \). We let \( F = \ker(D\pi) \) be the \textbf{fiberwise tangent bundle}: for \( x \in M^\alpha \), \( F(x) \subset T_xM^\alpha \) is the \( \dim(M) \)-dimensional subspace tangent to the fiber through \( x \).

Equip \( M^\alpha \) with a Riemannian metric.\(^7\) For convenience, we moreover assume the restriction of the metric to \( G \)-orbits coincides under push-forward by the projection \( \pi : M^\alpha \to \text{SL}(n, \mathbb{R})/\Gamma \) with the metric on \( \text{SL}(n, \mathbb{R})/\Gamma \) induced by the right-invariant metric on \( G \).

10.2. Fiberwise Lyapunov exponents. The \( G \)-action on \( M^\alpha \) induces an \( A \)-action on \( M^\alpha \). Let \( \mu \) be any ergodic, \( A \)-invariant Borel probability measure on \( M^\alpha \). The \( G \)-action (and hence the \( A \)-action) permutes the fibers of \( M^\alpha \) and hence the derivative of the \( G \)- and \( A \)-actions preserve the fiberwise tangent subbundle \( F \subset TM^\alpha \).

We identify \( A \simeq \mathbb{R}^{n-1} \) and equip \( A \simeq \mathbb{R}^{n-1} \) with a norm \( |\cdot| \). We may restrict Theorem 5.4 to the \( A \)-invariant subbundle \( F \subset TM^\alpha \) and obtain Lyapunov exponent functionals for the fiberwise derivative cocycle. We then obtain

1. linear functionals \( \lambda_{F,\mu}^1, \lambda_{F,\mu}^2, \ldots, \lambda_{F,\mu}^p : A \to \mathbb{R} \); and

2. a \( \mu \)-measurable, \( D\alpha \)-invariant splitting \( T_xM = \bigoplus_{i=1}^p E_i^F(x) \) defined for \( x \in A \) such that for every \( x \in A \) and \( v \in E_i^F(x) \setminus \{0\} \)

\[
\lim_{|n| \to \infty} \frac{\log \|D_x\alpha(n)(v)\| - \lambda_i(n)}{|n|} = 0;
\]

\(^7\)Note that if \( \Gamma \) is cocompact, \( M^\alpha \) is compact and all metrics are equivalent. In the case that \( \Gamma \) is not cocompact, some additional care is needed to ensure the metric is well behaved in the fibers. We will not discuss the technicalities of this case here.
implies that we have the following invariance principle first exploited in [wise Lyapunov exponents. In particular, for any $\beta \in A$ that each root $\beta$ general position with respect to every root $M$ to the fibers of $\Lambda$. Moreover, if every fiberwise Lyapunov exponent $\lambda_{\mu}^F$ is resonant with $\lambda_{i,\mu}^F$ for the action of $A$ on $\Gamma$ for any $A$-invariant measure. Each root $\beta^{i,j}$ has a corresponding root group $U^{i,j} \subset \text{SL}(n, \mathbb{R})$. Given an ergodic, $\mu$-invariant measure $\mu$ we also have fiberwise Lyapunov exponents $\lambda_{i,\mu}^F, \lambda_{2,\mu}^F, \ldots, \lambda_{p,\mu}^F : A \to \mathbb{R}$ for the restriction of the derivative of $A$-action on $(M^\alpha, \mu)$ to the fibers of $M^\alpha$.

Recall that both $\beta^{i,j}$ and $\lambda_{i,\mu}^F$ are linear functions on the vector space $A \simeq \mathbb{R}^{n-1}$. We will say that a root $\beta^{i,j}$ is resonant with a fiberwise Lyapunov exponent $\lambda_{i,\mu}^F$ of $\mu$ if they are positively proportional; that is, $\beta^{i,j}$ is resonant with $\lambda_{i,\mu}^F$ if there is a $c > 0$ with

$$\beta^{i,j} = c \lambda_{i,\mu}^F.$$ 

Otherwise we say that $\beta^{i,j}$ is not resonant with $\lambda_{i,\mu}^F$. We say that a root $\beta^{i,j}$ of $G$ is non-resonant if it is not resonant with any fiberwise Lyapunov exponent $\lambda_{i,\mu}^F$ for the ergodic, $\mu$-invariant measure $\mu$.

**Proposition 10.1.** Suppose $\mu$ is an ergodic, $\mu$-invariant measure on $M^\alpha$ projecting to the Haar measure on $\text{SL}(n, \mathbb{R})/\Gamma$ under the projection $\pi : M^\alpha \to \text{SL}(n, \mathbb{R})/\Gamma$.

Then, for every non-resonant root $\beta^{i,j}$, the measure $\mu$ is $U^{i,j}$-invariant.

**Remark 10.2.** Since each root $\beta^{i,j}$ is a non-zero functional on $A$, if a fiberwise exponent $\lambda_{i,\mu}^F$ is zero, then every root $\beta^{i,j}$ is not resonant with $\lambda_{i,\mu}^F$. If there are $p$ fiberwise Lyapunov exponents $\{\lambda_{i,\mu}^F, 1 \leq i \leq p\}$ or, more generally $p' \leq p$ coarse fiberwise Lyapunov exponents $\{\lambda_{i,\mu}^F, 1 \leq i \leq p'/\}$ or, then Proposition 10.1 implies that $\mu$ is invariant under all but $p'$ root subgroups $U^{i,j}$. Moreover, if every fiberwise Lyapunov exponent $\lambda_{i,\mu}^F$ is in general position with respect to every root $\beta^{i,j}$ then $\mu$ is automatically $G$-invariant.

**10.4. Coarse-Lyapunov Abramov-Rohlin Theorem; Proof of Proposition 10.1.** Note that each root $\beta^{i,j}$ of $\text{SL}(n, \mathbb{R})$ is also a Lyapunov exponent for the $A$-action on $(M^\alpha, \mu)$ (corresponding to vectors tangent to $U^{i,j}$ orbits in $M^\alpha$.) Let $\chi^{i,j}$ be the coarse Lyapunov exponent for the $A$-action on $(M^\alpha, \mu)$ containing $\beta^{i,j}$; that is, $\chi^{i,j}$ is the equivalence class of all Lyapunov exponents for the $A$-action on $(M^\alpha, \mu)$ that are positively proportional to $\beta^{i,j}$.

Let $\{\chi_{i,\mu}^F, 1 \leq i \leq p\}$ denote the fiberwise Lyapunov exponents. We have that

$$\chi^{i,j} = \{\beta^{i,j}\}$$ if $\beta^{i,j}$ is not resonant with any $\lambda_{i,\mu}^F$.

Otherwise, $\chi^{i,j}$ is the union of $\beta^{i,j}$ and all fiberwise Lyapunov exponents $\lambda_{i,\mu}^F : A \to \mathbb{R}$ that are positively proportional to $\beta^{i,j}$.
For $\mu$-a.e. $x \in M^\alpha$ there is a coarse Lyapunov manifold $W^{\chi^{i,j}}(x)$ through $x$. If $\chi^{i,j} = \{ \beta^{i,j} \}$ then for $x \in M^\alpha$, $W^{\chi^{i,j}}(x)$ is simply the $U^{i,j}$-orbit of $x$. Otherwise, $W^{\chi^{i,j}}(x)$ is a higher-dimensional manifold which intersects the fibers of $M^\alpha$ non-trivially. The partition of $(M^\alpha, \mu)$ into $W^{\chi^{i,j}}$-manifolds forms an $A$-invariant, measurable foliation.

If $\beta^{i,j}$ is resonant with some fiberwise exponent, let $\hat{\chi}^{i,j,F}$ be the coarse fiberwise Lyapunov exponent positively proportional to $\beta^{i,j}$; otherwise, let $\chi^{i,j,F}$ denote the zero functional. If $\chi^{i,j,F}$ is non-zero, for $\mu$-a.e. $x \in M^\alpha$ there is similarly a coarse Lyapunov manifold $W^{\chi^{i,j,F}}(x)$ through $x$. If $\chi^{i,j,F}$ is zero, let $W^{\chi^{i,j,F}}(x) = \{ x \}$. We then have that $W^{\chi^{i,j,F}}(x)$ is contained in the fiber through $x$ and that $W^{\chi^{i,j}}(x)$ is the $U^{i,j}$-orbit of $W^{\chi^{i,j,F}}(x)$.

For each $\chi^{i,j}$ and $a \in A$ with $\beta^{i,j}(a) > 0$ we can define a conditional entropy of $a$ conditioned on $\chi^{i,j}$-manifolds, denoted by $h_\mu(a \mid \chi^{i,j})$ as in Section 7. We can define a conditional entropy of $a$ conditioned on $\chi^{i,j,F}$-manifolds, denoted by $h_\mu(a \mid \chi^{i,j,F})$.

In this setting, we have the following “coarse-Lyapunov Abramov-Rohlin formula” which follows a specific case of Proposition 7.3.

**Theorem 10.3.** For any $a \in A$ with $\beta^{i,j}(a) > 0$,

$$h_\mu(a \mid \chi^{i,j}) = h_{\text{Haar}}(a \mid \beta^{i,j}) + h_\mu(a \mid \chi^{i,j,F}).$$

(10)

Above,

$$h_{\text{Haar}}(a \mid \beta^{i,j})$$

denotes the conditional entropy of translation by $a$ in $\text{SL}(n, \mathbb{R})/\Gamma$ conditioned along $U^{i,j}$-orbits in $\text{SL}(n, \mathbb{R})/\Gamma$.

**Proof of Theorem 10.3.** Note that, as the map $\pi: M^\alpha \to G/\Gamma$ is smooth, every coarse restricted root $\hat{\chi}$ for the action of $A$ on $G/\Gamma$ coincides with some coarse Lyapunov exponent $\chi$ for the action of $A$ on $(M^\alpha, \mu)$. Given $\chi \in \hat{\mathcal{L}}$, if $\chi$ is a restricted root on the factor $G/\Gamma$, then take $\hat{\eta}$ to be the point partition on $G/\Gamma$. Otherwise, write $\mathcal{W}^\chi$ and $\mathcal{W}_\chi$ respectively for the coarse Lyapunov foliations of $(M^\alpha, \mu)$ and $(G/\Gamma, \nu)$ of coarse Lyapunov exponent $\chi$. Also let $\zeta$ and $\hat{\eta}$ be respectively, the measurable partitions $\xi^{\mathcal{W}^\chi}$ of $(M, \mu)$ subordinate to $\mathcal{W}^\chi$, and $\xi^{\mathcal{W}_\chi}$ of $(G/\Gamma, \nu)$ subordinate to $\mathcal{W}_\chi$, both given by Lemma 6.3. Write $\eta = \pi^{-1}(\hat{\eta})$. Finally, let $\mathcal{F}^\chi = \mathcal{W}^\chi \lor \mathcal{F}$.

The partitions $\hat{\eta}$ and $\zeta$ satisfy

$$h_{\text{Haar}}(s, \hat{\eta}) = h_{\text{Haar}}(s \mid \chi), \text{ and } h_\mu(\hat{\alpha}(s), \zeta \lor \mathcal{F}) = h_\mu(\hat{\alpha}(s) \mid \mathcal{F}^\chi).$$

We have the following standard computation (c.f. [KRH, Lemma 6.1]):

$$h_\mu(\hat{\alpha}(s) \mid \chi) := h_\mu(\hat{\alpha}(s), \eta \lor \zeta)$$

$$\leq h_\mu(\hat{\alpha}(s), \eta) + h_\mu(\hat{\alpha}(s), \eta \lor \bigvee_{n \in \mathbb{Z}} \alpha(s^n)(\eta))$$

$$= h_{\text{Haar}}(s, \hat{\eta}) + h_\mu(\hat{\alpha}(s), \zeta \lor \mathcal{F})$$

$$= h_{\text{Haar}}(s \mid \chi) + h_\mu(\hat{\alpha}(s) \mid \mathcal{F}^\chi).$$
Now, as long as $\chi(s) > 0$, we have from the classical Abramov-Rokhlin lemma and the product structure of entropies that

$$h_\mu(\alpha(s)) = \sum_{\chi(s) > 0} h_\mu(\alpha(s) \mid \chi) \leq \sum_{\chi(s) > 0} h_{\text{Haar}}(s \mid \chi) + \sum_{\chi(s) > 0} h_\mu(\alpha(s) \mid F^\chi) = h_{\text{Haar}}(s) + h_\mu(\alpha(s) \mid F) = h_\mu(\alpha(s)).$$

Since entropies are non-negative quantities, it follows that

$$h_\mu(\alpha(s) \mid \chi) = h_{\text{Haar}}(s \mid \chi) + h_\mu(\alpha(s) \mid F^\chi)$$

for all $\chi \in \mathcal{L}$ with $\chi(s) > 0$. $\square$

The proof of Proposition 10.1 is a trivial consequence of Theorem 10.3.

**Proof of Proposition 10.1.** Given a root $\beta^{i,j}$ and $a \in A$ such that $\beta^{i,j}(a) > 0$ we may define a conditional entropy $h_\mu(a \mid \beta^{i,j})$ for the entropy of translation by $a$, conditioned on $U^{i,j}$-orbits in $M^\alpha$. From an appropriate version of the Margulis-Ruelle inequality (see Theorem 7.2(1) and (8)), for $a \in A$ with $\beta^{i,j}(a) > 0$ we have that

$$h_\mu(a \mid \beta^{i,j}) \leq \beta^{i,j}(a).$$ (11)

On the other hand, if $\beta^{i,j}$ is non-resonant then $\chi^{i,j,F}$ is the zero functional whence the term $h_\mu(a \mid \chi^{i,j,F})$ in (10) vanishes and $W^{\chi^{i,j}}(x)$ is simply the $U^{i,j}$-orbit of $x$ for every $x$. Hence, by Theorem 10.3,

$$h_\mu(a \mid \beta^{i,j}) = h_\mu(a \mid \chi^{i,j}) = h_{\text{Haar}}(a \mid \beta^{i,j}) = \beta^{i,j}(a).$$ (12)

From (11) and (12), we have that the conditional entropy $h_\mu(a \mid \beta^{i,j})$ attains is maximal possible value. In particular, from the invariance principle in Theorem 8.5(3) it follows that $\mu$ is $U^{i,j}$-invariant. $\square$

### 10.5. **Proof of Theorem 3.8.** Theorem 3.8 follows immediately from Proposition 10.1 in this case. First consider the case of actions on the circle $S^1$.

**Proof outline of Theorem 3.8 for actions on $S^1$.** For a $\Gamma$ action on $S^1$, Theorem 3.8 follows from the following 3 observations:

1. As an exercise left to the reader, we claim that there is a $\Gamma$-invariant probability measure on $M$ if and only if there is a $G$-invariant probability measure on $M^\alpha$.
2. Since $A$ is abelian (in particular amenable) and the space of probability measures on $M^\alpha$ projecting to the Haar measure on $\text{SL}(n, \mathbb{R})/\Gamma$ is non-empty, $A$-invariant, and weak-$*$ compact, the Krylov-Bogolyubov theorem will give an $A$-invariant probability measure $\mu$ on $M^\alpha$ projecting to the Haar measure on $\text{SL}(n, \mathbb{R})/\Gamma$. Moreover, since the Haar measure on $\text{SL}(n, \mathbb{R})/\Gamma$ is $\Gamma$-ergodic, we may assume $\mu$ is $\Gamma$-ergodic.

Since the fibers are one dimensional, there is a single fiberwise Lyapunov exponent $\lambda_F^\mu$ for the action of $A$ on $(M, \mu)$.

3. At most one root $\beta^{i_0,j_0}$ is resonant with the fiberwise Lyapunov exponent $\lambda_F$. All other roots $\beta^{i,j}$ are non-resonant. By Proposition 10.1, $\mu$ is invariant under the corresponding root groups $U^{i,j}$ for every $(i, j) \neq (i_0, j_0)$. However, the subgroups $A$ and $U^{i,j}$ for $(i, j) \neq (i_0, j_0)$ generate all of $G$. Hence $\mu$ is $G$-invariant. $\square$
Proof outline of Theorem 3.8 in higher dimensions. Steps (1) and (2) of the above outline continue to hold when when \( \alpha \) is a \( \Gamma \)-action on a \( d \)-dimensional manifold \( M \). In particular, we may find an ergodic, \( \Lambda \)-invariant measure \( \mu \) projecting to the Haar measure on \( \text{SL}(n, \mathbb{R})/\Gamma \). The tangent bundle \( F \) along the \( M \)-fibers is now \( d \)-dimensional and therefore splits into at most \( d \) Fiberwise Lyapunov exponents \( \lambda_1^\mu, \ldots, \lambda_k^\mu \), \( k \leq d \). By Proposition 10.1, if \( \beta_{ij} \) is not resonant to any \( \lambda_m^\mu \), then \( \mu \) is \( U_{ij} \)-invariant.

When \( d < n - 1 \), \( \mu \) is invariant under \( \Lambda \) and all but at most \( n - 2 \) root subgroups \( U_{ij} \). However, the largest proper closed subgroup containing \( \Lambda \) and the maximal number of subgroups \( U_{ij} \)’s is conjugate to the maximal parabolic subgroup

\[
\begin{pmatrix}
* & * & \cdots & * \\
0 & * & \cdots & * \\
0 & \vdots & \ddots & \vdots \\
0 & * & \cdots & *
\end{pmatrix}
\]

which misses as many as \( n - 1 \) root subgroups.

Therefore, \( \Lambda \) and the root subgroups that stabilize \( \mu \) generate all of \( G \). Thus \( \mu \) is \( G \)-invariant. \( \square \)

Part 3. Zimmer’s conjecture: proof of Theorem 3.9

11. PROOF SKETCH OF THEOREMS 3.9

We outline the proof of Theorems 3.9 for the case of \( C^\infty \) actions. That is, for \( n \geq 3 \) we consider a cocompact lattice \( \Gamma \) in \( \text{SL}(n, \mathbb{R}) \) and show that every homomorphism \( \alpha : \Gamma \to \text{Diff}^1(M) \) has finite image when

1. \( M \) is a compact manifold of dimension at most \((n - 2)\), or
2. \( M \) is a compact manifold of dimension at most \((n - 1)\), and \( \alpha \) preserves a volume form \( \text{vol} \).

The outline consists of 3 steps.

Step 1: Subexponential growth. It is a standard exercise to show that discrete groups acting cocompactly on a metric space \((X, d)\) are finitely generated; consequently, cocompact lattices are finitely generated. More generally, it is a classical fact that all lattices \( \Gamma \subset \text{SL}(n, \mathbb{R}) \) are finitely generated. Fix a finite symmetric generating set \( S \) for \( \Gamma \). Given \( \gamma \in \Gamma \), let \( |\gamma| = |\gamma|_S \) denote the word-length of \( \gamma \) relative to this generating set; that is,

\[
|\gamma| = \min\{k : \gamma = s_k \cdots s_1, s_i \in S\}.
\]

Note that if we replace the finite generating set \( S \) by \( S' \), there is a uniform constant \( C \) such that the word lengths are uniformly distorted:

\[
|\gamma|_{S'} \leq C|\gamma|_S.
\]

Thus all definitions below will be independent of the choice of \( S \).

Equip \( M \) with a Riemannian metric.

**Definition 11.1.** We say that an action \( \alpha : \Gamma \to \text{Diff}^1(M) \) has uniform subexponential growth of derivatives if for every \( \varepsilon > 0 \) there is a \( C = C_\varepsilon \) such that

\[
\sup_{x \in M} \|D_x \alpha(\gamma)\| \leq Ce^{\varepsilon |\gamma|}.
\]

The following is the main theorem from in [BFH].
Theorem 11.2 ([BPH, Theorem 2.8]). For $n \geq 3$, let $\Gamma \subset \text{SL}(n, \mathbb{R})$ be a cocompact lattice. Let $\alpha : \Gamma \to \text{Diff}^2(M)$ be an action. Suppose that either

1. $\dim(M) \leq n - 2$, or
2. $\dim(M) = n - 1$ and $\alpha$ preserves a smooth volume.

Then $\alpha$ has uniform subexponential growth of derivatives.

Step 2: Strong property (T). Assume $\alpha : \Gamma \to \text{Diff}^\infty(M)$ is an action by $C^\infty$ diffeomorphisms.\(^8\) The action $\alpha$ of $\Gamma$ on $M$ induces an action $\alpha_\#$ of $\Gamma$ on tensor powers of the cotangent bundle of $M$ by pull-back over the opposite\(^9\) action: given $\omega \in (T^*M)^\otimes k$,

$$\alpha_\#(\gamma) \omega = \alpha(\gamma^{-1})^* \omega;$$

that is, if $v_1, \ldots, v_k \in T_xM$ then

$$\alpha_\#(\gamma) \omega(x)(v_1, \ldots, v_k) = \omega(x)(D_x\alpha(\gamma^{-1})v_1, \ldots, D_x\alpha(\gamma^{-1})v_k).$$

In particular, we obtain an action of $\Gamma$ on Riemannian metrics which naturally sit inside $S^2(T^*M)$, the set of all symmetric 2-forms on $M$. Note that $\alpha_\#$ preserves $C^\ell(S^2(T^*M))$, the subspace of all $C^\ell$ sections of $S^2(T^*M)$ for any $\ell \in \mathbb{N}$.

Fix a volume form $\text{vol}$ on $M$. The norm on $TM$ induced by a background Riemannian metric induces a norm on each fiber of $S^2(T^*M)$. We then obtain a natural notion of measurable and integrable sections of $S^2(T^*M)$ with respect to $\text{vol}$. Let $\mathcal{H}^k = W^{2,k}(S^2(T^*M))$ be the Sobolev space of symmetric 2-forms whose weak $C^\ell$-derivatives are bounded with respect to the $L^2(\text{vol})$-norm for $0 \leq \ell \leq k$. Then $\mathcal{H}^k$ is a Hilbert space. Let $\| \cdot \|_{\mathcal{H}^k}$ denote the corresponding Sobolev norm. Working in local coordinates, the Sobolev embedding theorem gives that

$$\mathcal{H}^k \subset C^\ell(S^2(T^*M))$$

as long as

$$\ell < k - \dim(M)/2.$$ 

In particular, for $k$ sufficiently large, an element $\omega$ of $\mathcal{H}^k$ gives a $C^\ell$ section of $S^2(T^*M)$ which will be a $C^\ell$ Riemannian metric on $M$ if it is positive definite.

The action $\alpha_\#$ is a representation of $\Gamma$ by bounded operators on $\mathcal{H}^k$. Using theorem 11.2 we obtain strong control on the norm grow of the induced representation $\alpha_\#$. In particular, we obtain subexponential norm growth:

Lemma 11.3. Let $\alpha : \Gamma \to \text{Diff}^{k+1}(M)$ have uniform subexponential growth of derivatives. Then, for all $\varepsilon' > 0$ there is $C > 0$ such that

$$\|\alpha_\#(\gamma)\|_{\mathcal{H}^k} \leq C e^{\varepsilon' |\gamma|}.$$ 

The proof of Lemma 11.3 follows from the chain rule and a standard computation which shows that the growth of higher-order derivatives is controlled by the growth of the first derivatives.

We use the main result from [dLdLs]: cocompact lattices $\Gamma$ in higher-rank simple Lie groups (such as $\text{SL}(n, \mathbb{R})$ for $n \geq 3$) satisfy Lafforgue’s strong Banach property (T) first introduced in [Laf]. Strong Banach property (T) considers representations $\pi$ of $\Gamma$ by bounded operators on certain\(^10\) Banach spaces $E$. If such representations have sufficiently

---

\(^8\)For $C^\ell$ actions, one replaces the Hilbert Sobolev spaces $W^{2,k}(S^2(T^*M))$ below with appropriate Banach Sobolev spaces $W^{p,1}(S^2(T^*M))$ and verifies such spaces are of the type $\mathcal{E}_{10}$ considered in [dLdLs].

\(^9\)Note that since pullbacks are contravariant, in order for $\alpha_\#$ to be an action of $\Gamma$ on $(T^*M)^\otimes k$ we use the opposite action. Alternatively, one can use the pullbacks over the original action $\alpha$ to obtain an action of $\Gamma_{op}$.

\(^10\)of type $\mathcal{E}_{10}$
slow exponential norm growth, then there exists sequence of averaging operators $p_n$ converging to a projection $p_\infty$ such that for any vector $v \in E$, the limit $p_\infty(v)$ is $\pi$-invariant. In the case that $E$ is a Hilbert space (which we may assume when $G$ is an action by $C^\infty$ diffeomorphisms) we have the following formulation whose hypotheses are satisfied by Lemma 11.3.

**Theorem 11.4** ([dLdlS]). Let $\mathcal{H}$ be a Hilbert space and for $n \geq 3$, let $\Gamma$ be a cocompact lattice in $\text{SL}(n, \mathbb{R})$.

There exists $\varepsilon > 0$, such that for any representation $\pi: \Gamma \to B(\mathcal{H})$, if there exists $C_\varepsilon > 0$ such that

$$\|\pi(g)\| \leq C_\varepsilon e^{\varepsilon|\gamma|}$$

then there exists a sequence of averaging operators $p_n = \sum w_i \pi(\gamma_i)$ in $B(\mathcal{H})$—where $w_i > 0$, $\sum w_i = 1$, and $w_i = 0$ for every $\gamma_i \in \Gamma$ of word-length larger than $n$—such that for any vector $v \in \mathcal{H}$ the sequence $v_n = p_n(v) \in \mathcal{H}$ converges to an invariant vector $v^* = p_\infty(v)$.

Moreover the convergence is exponentially fast: there exist $0 < \lambda < 1$ and a $C_\lambda$ so that

$$\|v_n - v^*\| \leq C_\lambda^n \|v\|.$$

Consider an arbitrary $C^\infty$ Riemannian metric $g$. We have $g \in \mathcal{H}^k$. As averages of finitely many Riemannian metrics are still Riemannian metrics we have $g_n := p_n(g)$ is positive definite for every $n$. In particular, the limit $g_\infty = p_\infty(g)$ is in the closed cone of positive symmetric 2-tensors. Taking $k$ sufficiently large we have that $g_\infty$ is $C^k$; in particular, $g_\infty$ is continuous, everywhere defined, and positive everywhere. We only need to confirm that $g_\infty$ is non-degenerate, i.e. positive-definite.

Given $x \in M$ and a unit vector $\xi \in T_x M$, for any $\varepsilon > 0$ we have from Definition 11.1 that there is a $C_\varepsilon > 0$ such that

$$p_n(g)(\xi, \xi) = \sum w_i \alpha_\#(\gamma_i)(\xi, \xi)$$

$$= \sum w_i (D_\alpha(\gamma_i \in \xi) \xi, D_\alpha(\gamma_i \in \xi) \xi)$$

$$\geq \frac{1}{C_\varepsilon} e^{-2\varepsilon n}.$$

On the other hand, from the exponential convergence in Theorem 11.4 we have

$$|p_n(g)(\xi, \xi) - p_\infty(g)(\xi, \xi)| \leq C_\lambda^n.$$

Thus

$$p_\infty(g)(\xi, \xi) \geq \frac{1}{C_\varepsilon^2} e^{-2\varepsilon n} - C_\lambda^n$$

for all $n \geq 0$. Taking $\varepsilon > 0$ sufficiently small we can ensure that

$$C_\varepsilon^2 e^{-2\varepsilon n} < \frac{1}{C_\lambda} \lambda^{-n}$$

for all sufficiently large $n$. Then, for all sufficiently large $n$ we have

$$\frac{1}{C_\varepsilon^2} e^{-2\varepsilon n} > C_\lambda \lambda^n$$

and thus

$$p_\infty(g)(\xi, \xi) > 0.$$

Note that $p_\infty(g)(\xi, \xi) > 0$ for every $x \in M$ and any $\xi \in T_x M$. Thus $g_\infty = p_\infty(g)$ is positive definite and hence $g_\infty$ is an $\alpha$-invariant $C^k$ Riemannian metric.
Step 3: Margulis superrigidity with compact codomain. From Steps 1 and 2 we have that any action \( \alpha : \Gamma \to \text{Diff}^\infty(M) \) preserves a \( C^f \) Riemannian metric \( g \). In the general case of \( C^2 \)-actions, we have that any action \( \alpha : \Gamma \to \text{Diff}^2(M) \) preserves a continuous Riemannian metric \( g \). Thus we have

\[
\alpha : \Gamma \to \text{Isom}_g^2(M) \subset \text{Diff}^2(M).
\]

Let \( \dim(M) = m \). The group \( \text{Isom}_g^2(M) \) is of isometries of a continuous Riemannian metric is a compact Lie group with

\[
\dim(\text{Isom}_g^2(M)) \leq \frac{m(m+1)}{2}.
\]

With \( K = \text{Isom}_g^2(M) \subset \text{Diff}^2(M) \) we thus obtain a compact-valued representation \( \alpha : \Gamma \to K \). By equation (13), if \( m < \frac{1}{2}\sqrt{8n^2 - 7} - \frac{1}{2} \) then \( \dim(\text{su}(n)) = n^2 - 1 > \dim(K) \) and so conclusion (2) of Theorem 4.4 is impossible. In particular, if \( m \leq n - 1 \) then conclusion (2) of Theorem 4.4 is impossible whence we conclude that the image

\[
\alpha(\Gamma) \subset K = \text{Isom}_g^2(M) \subset \text{Diff}^2(M)
\]

is finite.

Summarizing the above arguments, we obtain the following.

**Theorem 11.5.** For \( n \geq 3 \), let \( \Gamma \subset \text{SL}(n, \mathbb{R}) \) be a cocompact lattice. Let \( \alpha : \Gamma \to \text{Diff}^2(M) \) be an action with uniform subexponential growth of derivatives. Then, if

\[
\dim(M) < \frac{1}{2}\sqrt{8n^2 - 7} - \frac{1}{2},
\]

the image \( \alpha(\Gamma) \) is finite.

12. Proof outline of Theorem 11.2

To establish Theorem 3.9, from the discussion in Section 11 it is enough to establish Theorem 11.2, the uniform subexponential growth of derivatives for the action \( \alpha \). We outline the proof the proof of Theorem 11.2(1) in the case that \( \Gamma \) is a cocompact lattice in \( \text{SL}(n, \mathbb{R}) \).

12.1. Setup for proof. For \( n \geq 3 \), let \( \Gamma \subset \text{SL}(n, \mathbb{R}) \) be a cocompact lattice. Let \( M \) be a compact manifold with \( \dim(M) \leq n - 2 \) and let \( \alpha : \Gamma \to \text{Diff}^2(M) \) an action. We recall the following constructions from the proof of Theorem 3.8:

1. The manifold \( M^\alpha = (\text{SL}(n, \mathbb{R}) \times M)/\Gamma \) is the suspension space introduced in Section 10.1. \( M^\alpha \) is fiber bundle over \( \text{SL}(n, \mathbb{R})/\Gamma \) with fibers diffeomorphic to \( M \). Moreover, \( M^\alpha \) and \( \text{SL}(n, \mathbb{R})/\Gamma \) have a natural (left) \( \text{SL}(n, \mathbb{R}) \)-actions and the projection \( \pi : M^\alpha \to \text{SL}(n, \mathbb{R})/\Gamma \) intertwines these \( G \)-actions.
2. \( \Lambda \subset \text{SL}(n, \mathbb{R}) \) denotes the subgroup of diagonal matrices with positive entries. We have \( \Lambda \simeq \mathbb{R}^{n-1} \) which is a higher-rank free abelian group if \( n \geq 3 \).
3. Given an ergodic, \( \Lambda \)-invariant Borel probability measure \( \mu \) on \( M^\alpha \) we have fiberwise Lyapunov exponents

\[
\lambda^F_{1,\mu}, \ldots, \lambda^F_{p,\mu} : \Lambda \to \mathbb{R}
\]

for the restriction of the the derivative of the \( \Lambda \)-action to the fibers of \( M^\alpha \) introduced in Section 10.2.
4. \( \beta^{i,j} : \Lambda \to \mathbb{R} \) are the roots of \( \text{SL}(n, \mathbb{R}) \) and \( U^{i,j} \) are the corresponding root subgroups introduced in Section 9.1.
12.2. Two key propositions. The proof of Theorem 11.2 is by contradiction and follows from the following two propositions.

Proposition 12.1. Suppose that $\alpha : \Gamma \to \text{Diff}(M)$ fails to have uniform subexponential growth of derivatives. Then there exists a Borel probability measure $\mu'$ on $M^\alpha$ such that

1. $\mu'$ is $A$-invariant and ergodic;
2. there exists a non-zero fiberwise Lyapunov exponent $\lambda^F_{j,\mu'} : A \to \mathbb{R}$.

The proof of Proposition 12.1 is analogous to the proof of Proposition 5.2. We won’t include it here but it can be found in [BFH, Section 4].

The measure $\mu'$ in Proposition 12.1 projections to an ergodic $A$-invariant measure on $\text{SL}(n, \mathbb{R})/\Gamma$. If $\mu'$ projected to the Haar measure on $\text{SL}(n, \mathbb{R})/\Gamma$ then, as explained below, we would be done. However, it is known that there exist ergodic $A$-invariant measures on $\text{SL}(n, \mathbb{R})/\Gamma$ that are not the Haar measure.\footnote{In fact, for certain lattices $\Gamma$ there exist ergodic $A$-invariant measures on $\text{SL}(n, \mathbb{R})/\Gamma$ that have positive entropy for some element of $A$ as shown by M. Rees; see [EK1, Section 8].}

By carefully averaging the measure $\mu'$ along root subgroups $U^{i,j}$ and applying Ratner’s measure classification theorem (see [Rat]) to the projection to $\text{SL}(n, \mathbb{R})/\Gamma$ we obtain the following.

Proposition 12.2. Suppose there exists an ergodic, $A$-invariant measure $\mu'$ on $M^\alpha$ with a non-zero fiberwise Lyapunov exponent $\lambda^F_{j,\mu'}$. Then there exists a Borel probability measure $\mu'$ on $M^\alpha$ such that

1. $\mu$ is $A$-invariant and ergodic;
2. there exists a non-zero fiberwise Lyapunov exponent $\lambda^F_{j,\mu'} : A \to \mathbb{R}$;
3. $\mu$ projects to the Haar measure on $\text{SL}(n, \mathbb{R})/\Gamma$.

Propositions 12.1 and 12.2 hold in full generality; they do not depend on the dimension of $M$ compared to the rank of $\text{SL}(n, \mathbb{R})$. The contradiction in the proof of Theorem 11.2 arises by applying Zimmer’s cocycle superrigidity for the fiberwise derivative cocycle and using that the dimension of the fiber (which is the dimension of $M$) is small compared to the rank of $G$.

12.3. Proof of Theorem 11.2. We deduce Theorem 11.2(1) from Propositions 12.1 and 12.2 and Theorem 4.2. Theorem 11.2(2) follows from a similar argument using that the sum of fiberwise Lyapunov exponents is zero.

Proof of Theorem 11.2(1). Since the measure $\mu$ in Proposition 12.2 projects to the Haar measure on $\text{SL}(n, \mathbb{R})/\Gamma$ we may employ the invariance principle in Proposition 10.1. In particular, if $\dim(M) \leq n - 2$ there are at most $n - 2$ roots $\beta^{i,j}$ that are resonant with the fiberwise Lyapunov exponents $\{\lambda^F_{i,\mu}\}$. It follows from Proposition 10.1 that $\mu$ is invariant by $A$ and all-but $(n - 2)$ root subgroups $U^{i,j}$. In particular, $\mu$ is invariant by a subgroup $H \subset \text{SL}(n, \mathbb{R})$ with codimension at most $n - 2$. However, from the structure theory\footnote{For instance, one can show that $H$ is necessarily parabolic. However, the minimal codimension of a non-trivial parabolic subgroup of $\text{SL}(n, \mathbb{R})$ is $(n - 1)$.} of $\text{SL}(n, \mathbb{R})$ it follows that there are no non-trivial subgroups $H \subset \text{SL}(n, \mathbb{R})$ with codimension less than $n - 1$. It follows that the measure $\mu$ in Proposition 12.2 is $G$-invariant.

Recall we write $\pi : M^\alpha \to \text{SL}(n, \mathbb{R})/\Gamma$ for the natural projection and let $F$ be the fiberwise tangent bundle; that is $F$ is sub-vector-bundle of $TM^\alpha$ given by $F = \ker D\pi$. We may then then apply Zimmer cocycle superrigidity theorem, Theorem 4.2, to the fiberwise derivative cocycle $A(g, x) = D_x g|_{F(x)}$ of the $\text{SL}(n, \mathbb{R})$-action on $M^\alpha$. Since
the fibers have dimension at most \( n - 1 \), it follows that the fiberwise derivative cocycle \( A(g, x) = D_x g|_{F(x)} \) is cohomologous to a compact-valued cocycle: there is a compact group \( K \subset \text{SL}(d, \mathbb{R}) \) and measurable \( \Phi: M^\alpha \to \text{GL}(d, \mathbb{R}) \) such that
\[
\Phi(g \cdot x) D_x g|_{F(x)} \Phi(x)^{-1} \in K.
\]
By Poincaré recurrence to sets on which the norm and conorm of \( \Phi \) are bounded, it follows for any \( g \in G \) and \( \varepsilon > 0 \) that the set of \( x \in M^\alpha \) such that
\[
\|D_x g^n|_{F(x)}\| \geq e^{\varepsilon n}
\]
has \( \mu \)-measure zero. This however, contradicts the existence of non-zero fiberwise Lyapunov exponent for \( \mu \). This contradiction completes the proof of Theorem 11.2.

\[
\square
\]

The contradiction now follow from Zimmer’s cocycle superrigidity which we explain in the next section.

13. Discussion of the proof of Proposition 12.2

We outline the main steps in the proof of Proposition 12.2. The proof follows from averaging the measure \( \mu' \) appearing in Proposition 12.1 along certain unipotent root groups \( U^{i,j} \) while controlling the fiberwise Lyapunov exponents for the action of various elements \( s \in A \).

13.1. Averaging measures on \( M^\alpha \). Let \( H = \{h^t : t \in \mathbb{R}\} \) be a 1-parameter subgroup of \( \text{SL}(n, \mathbb{R}) \). Given a measure \( \mu \) on \( M^\alpha \) and \( T \geq 0 \) we define
\[
H^T * \mu := \frac{1}{T} \int_0^T (h^t)_* \mu \, dt
\]
to be measure obtained by averaging the translates of \( \mu \) over the interval \( [0, T] \).

Let \( s \in A \). Given any \( s \)-invariant measure \( \mu \) on \( M^\alpha \), the \textbf{average top fiberwise Lyapunov exponent} of \( s \) with respect to \( \mu \) is
\[
\lambda_{\text{top}}^F(s, \mu) = \inf_{n \to \infty} \frac{1}{n} \int \log \|D_x (s^n)|_{F}\| \, d\mu(x).
\]
Note that if \( \mu \) is moreover \( A \)-invariant and \( A \)-ergodic with fiberwise Lyapunov exponents \( \lambda_{i,\mu}^F, \ldots, \lambda_{p,\mu}^F: A \to \mathbb{R} \) then
\[
\lambda_{\text{top}}^F(s, \mu) = \max_{1 \leq i \leq p} \lambda_{i,\mu}^F(s).
\]

We have the following facts which we invoke throughout.

**Claim 13.1.** Let \( s \in A \) and let \( \mu \) be an \( s \)-invariant measure on \( M^\alpha \). Let \( H = \{h^t, t \in \mathbb{R}\} \) be a one-parameter group contained in the centralizer of \( s \) in \( G \).

1. The measure \( H^T * \mu \) is \( s \)-invariant for every \( T \geq 0 \).
2. Any weak-* limit point of \( \{H^T * \mu : T \geq 0\} \) is \( s \)-invariant.
3. Any weak-* limit point of \( \{H^T * \mu : T \geq 0\} \) is \( H \)-invariant.
4. \( \lambda_{\text{top}}^F(s, H^T * \mu) = \lambda_{\text{top}}^F(s, \mu) \) for every \( T \geq 0 \).
5. If \( \mu' \) is a weak-* limit point of \( \{H^T * \mu : T \geq 0\} \) then
\[
\lambda_{\text{top}}^F(s, \mu') \geq \lambda_{\text{top}}^F(s, \mu).
\]

(1) is clear from definition and (2) follows as the set of \( s \)-invariant measures is closed. (3) follows from the proof of the Krylov-Bogolyubov theorem. (4) is a standard computation which follows from the compactness of \( M^\alpha \). (5) follows from the well-known fact
that the average top Lyapunov exponent is upper-semicontinuous on the set of \(s\)-invariant measures.

Similar properties as in Claim 13.1 hold when averaging \(s\)-invariant measures against F"olner sets in higher-rank abelian groups, nilpotent groups, and more general amenable groups.

13.2. Averaging measures on \(\text{SL}(n, \mathbb{R})/\Gamma\). When averaging measures on \(\text{SL}(n, \mathbb{R})/\Gamma\) by 1-parameter unipotent subgroups we obtain additional properties of the limiting measures with follow as consequences of Ratner’s measure classification and equidistribution theorems for unipotent flows \([\text{Rat}]\).

Claim 13.2. Let \(\hat{\mu}\) be a Borel probability measure on \(\text{SL}(n, \mathbb{R})/\Gamma\). For each 1-parameter subgroup \(U^{i,j}\)

\[U^{i,j} \ast \hat{\mu} := \lim_{T \to \infty} \{(U^{i,j})^T \ast \hat{\mu} : T \geq 0\}\]

exists;

(2) if \(\hat{\mu}\) is \(A\)-invariant, so is \(U^{i,j} \ast \hat{\mu}\);

(3) if \(\hat{\mu}\) is \(A\)-invariant and \(A\)-ergodic, the measure \(U^{i,j} \ast \hat{\mu}\) is \(A\)-ergodic;

(4) if \(\hat{\mu}\) is \(A\)-invariant and \(U^{i,j}\)-invariant then \(\hat{\mu}\) is \(U^{j,i}\)-invariant.

Claim 13.2(1) follows from Ratner’s measure classification and equidistribution theorems for unipotent flows. Claim 13.2(2) follows from the fact that \(A\) normalized \(U^{i,j}\) and that the limit in Claim 13.2(1) exists and is hence unique. Claim 13.2(3) is an application of the pointwise ergodic theorem. Claim 13.2(4) is stated as Theorem 9 in \([\text{Rat}]\).

13.3. Proof sketch of Proposition 12.2 for \(\text{SL}(3, R)\). To simplify ideas, we outline the proof of Proposition 12.2 assuming \(\Gamma\) is a cocompact lattice in \(\text{SL}(3, \mathbb{R})\). We perform two averaging procedures on the measure \(\mu'\) given by Proposition 12.1 to obtain the measure guaranteed by Proposition 12.2.

Proof sketch of Proposition 12.2 for \(\text{SL}(3, R)\). Take \(\mu_0 = \mu'\) to be the measure guaranteed by Proposition 12.1 with non-zero fiber exponent

\[\lambda^F_{j,\mu_0} : A \to \mathbb{R}, \quad \lambda^F_{j,\mu_0} \neq 0.\]

First averaging. Consider the elements

\[s = \text{diag}(\frac{1}{2}, 2, 2) \quad \text{and} \quad \overline{s} = (2, 2, \frac{1}{2})\]

of \(A \subset \text{SL}(3, \mathbb{R})\). Note that \(s\) and \(\overline{s}\) are linearly independent and hence form a basis for \(A \simeq \mathbb{R}^2\). As the linear functional \(\lambda^F_{j,\mu_0}\) is non-zero, either

\[\lambda^F_{j,\mu_0}(s) \neq 0 \quad \text{or} \quad \lambda^F_{j,\mu_0}(\overline{s}) \neq 0.\]

Take \(s_0\) to be either \(s, s^{-1}, \overline{s}, \text{ or } \overline{s}^{-1}\) so that \(\lambda^F_{j,\mu_0}(s_0) > 0\). Without loss of generality we may assume that

\[\lambda^F_{j,\mu_0}(s) \neq 0.\]

Take \(s_0\) to be either \(s\) or \(s^{-1}\) so that \(\lambda^F_{j,\mu_0}(s_0) > 0\).

Consider the 1-parameter subgroup

\[U^{2,3} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}.\]
We check that $U^{2,3}$ commutes with $s_0$. Let $\mu_1$ be any weak-$*$ limit point of $\{(U^{2,3})^T \ast \mu : T \geq 0\}$. From Claim 13.1, $\mu_1$ is $s$-invariant and $\lambda_{\text{top}}^F(s_0, \mu_1) \geq \lambda_{\text{top}}^F(s_0, \mu_0)$.

We now average $\mu_1$ over a Følner sequence in $A$: identifying $A \simeq \mathbb{R}^2$ let

$$A^T \ast \mu_1 := \int_0^T \int_0^T (t_1, t_2) \ast \mu_1 \, dt(t_1, t_2).$$

Let $\mu_2$ be any weak-$*$ limit point of $\{A^T \ast \mu_1 : T \geq 0\}$. Then, from facts analogous to those in Claim 13.1, $\mu_2$ is $A$-invariant and

$$\lambda_{\text{top}}^F(s_0, \mu_2) \geq \lambda_{\text{top}}^F(s_0, \mu_1) > 0.$$

Note that $\mu_2$ may no longer be $U^{2,3}$-invariant.

We investigate the projection of each measure $\mu_0, \mu_1,$ and $\mu_2$ to $\text{SL}(3, \mathbb{R})/\Gamma$. For each $j$, we denote by $\hat{\mu}_j = \pi_\ast(\mu_1)$ the image of $\mu_1$ under the projection $M^\alpha \to \text{SL}(3, \mathbb{R})/\Gamma$.

Observe that $\hat{\mu}_1 = U^{2,3} \ast \hat{\mu}$ is $U^{2,3}$-invariant. Since $\hat{\mu}_0$ was $A$-invariant, from Claim 13.2(2) we have that $\hat{\mu}_1$ is $A$-invariant and it follows that $\hat{\mu}_1 = \hat{\mu}_2$ so $\hat{\mu}_2$ is $U^{2,3}$-invariant and $A$-invariant. From Claim 13.2(4), $\hat{\mu}_2$ is invariant under the group $\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \}$ generated by $A$, $U^{2,3}$ and $U^{3,2}$ in $\text{SL}(3, \mathbb{R})$. Moreover, $\hat{\mu}_0$ was $A$-ergodic, from Claim 13.2(3) the measure $\mu_1 = \hat{\mu}_2$ is $A$-ergodic.

Returning to $M^\alpha$, as $\lambda_{\text{top}}^F(s_0, \mu_2) > 0$ and as $\hat{\mu}_2$ is $A$-ergodic, we may replace $\mu_2$ with an $A$-ergodic component $\mu'_2$ of $\mu_2$ such that

1. $\lambda_{\text{top}}^F(s_0, \mu'_2) > 0$, and
2. the projection of $\mu'_2$ to $\text{SL}(3, \mathbb{R})/\Gamma$ is $\hat{\mu}_2$.

Let $\lambda_{1, \mu'_2}^F : A \to \mathbb{R}$ denote the fiberwise Lyapunov exponents for the $A$-invariant, $A$-ergodic measure $\mu'_2$, then $0 < \lambda_{1, \mu'_2}^F(s_0) = \lambda_{\text{top}}^F(s_0, \mu'_2)$ for some $1 < j' < p'$ whence some fiberwise Lyapunov exponent $\lambda_{j', \mu'_2}^F : A \to \mathbb{R}$ is a non-zero linear functional.

**Second averaging.** Consider now the elements $s = (2, 2, \frac{1}{4})$ and $\pi = (2, \frac{1}{4}, 2)$ in $A$.

Again, either

$$\lambda_{j', \mu'_2}^F(s) \neq 0 \quad \text{or} \quad \lambda_{3, \mu'_2}^F(\pi) \neq 0.$$

**Case 1: $\lambda_{j', \mu'_2}^F(s) \neq 0$.** Take $s_1 = s$ or $s_1 = s^{-1}$ so that $\lambda_{j', \mu'_2}^F(s_1) \neq 0$. Consider the one-parameter group $U^{1,2}$ which commutes with $s_1$. As above, any weak-$*$ limit point $\mu_3$ of $\{(U^{1,2})^T \ast \mu'_2 : T \geq 0\}$ is $s_1$-invariant, with

$$\lambda_{\text{top}}^F(s_1, \mu_3) = \lambda_{\text{top}}^F(s_1, \mu_3) \geq \lambda_{\text{top}}^F(s_1, \mu_2) > 0.$$

Let $\mu_4$ be any weak-$*$ limit point of $\{A^T \ast \mu_1 : T \geq 0\}$. Then $\mu_4$ is $A$-invariant and

$$\lambda_{\text{top}}^F(s_1, \mu_4) \geq \lambda_{\text{top}}^F(s_1, \mu_3) > 0.$$

We claim that the projection $\hat{\mu}_4$ of $\mu_4$ to $\text{SL}(3, \mathbb{R})/\Gamma$ is the Haar measure. Since the groups $U^{1,2}$ and $U^{3,2}$ commute and since $\mu_2$ was $U^{3,2}$-invariant, it follows that $\hat{\mu}_3$ is $U^{3,2}$-invariant. Also, since $\hat{\mu}_2$ was $A$-invariant, Claim 13.2(2) shows that $\hat{\mu}_3$ is $A$-invariant. Thus $\hat{\mu}_3 = \hat{\mu}_4$ and $\hat{\mu}_4$ is also invariant under the actions of $A$, $U^{1,2}$, and $U^{3,2}$. By Claim 13.2(4) it follows that $\hat{\mu}_4$ is invariant under the groups $U^{2,1}$ and $U^{2,3}$; in particular $\hat{\mu}_4$ is
invariant under the groups
\[
\left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \begin{pmatrix} * & * & 0 \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix} \right\}.
\]
These two groups generate all of $\text{SL}(3, \mathbb{R})$, and hence $\hat{\mu}_4$ is the Haar measures.

**Case 2:** $\lambda^{F}_{\mu', \mu''}(\Sigma) \neq 0$. Take $s_1 = \Sigma$ or $s_1 = \Sigma^{-1}$ so that $\lambda^{F}_{\mu', \mu''}(s_1) \neq 0$. Consider the one-parameter group $U^{1,3}$ which commutes with $s_1$. As above, any weak-* limit point $\mu_3$ of $\{(U^{1,3})^n \ast \mu_2 : T \geq 0\}$ is $s_1$-invariant, with
\[
\lambda^{F}_{\text{top}}(s_1, \mu_3) \geq \lambda^{F}_{\text{top}}(s_1, \mu_2) > 0.
\]
Let $\mu_4$ be any weak-* limit point of $\{A^n \ast \mu_3 : T \geq 0\}$. Then $\mu_4$ is $A$-invariant and
\[
\lambda^{F}_{\text{top}}(s_1, \mu_4) \geq \lambda^{F}_{\text{top}}(s_1, \mu_3) > 0.
\]
Again, we claim that $\hat{\mu}_4$ is the Haar measure. Since the groups $U^{1,3}$ and $U^{2,3}$ commute, it follows that $\hat{\mu}_3$ is $U^{2,3}$-invariant. Also, since $\hat{\mu}_2$ was $A$-invariant, Claim 13.2(2) shows that $\hat{\mu}_3$ is $A$-invariant. Thus $\hat{\mu}_3 = \hat{\mu}_4$ and $\hat{\mu}_4$ is also invariant under the actions of $A$, $U^{1,3}$ and $U^{2,3}$. By Claim 13.2(4) it follows that $\hat{\mu}_4$ is invariant under the groups
\[
\left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \begin{pmatrix} * & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix} \right\}.
\]
Again, these two groups generate all of $\text{SL}(3, \mathbb{R})$, and hence $\hat{\mu}_4$ is the Haar measure.

**Completion of proof.** In either Case 1 or Case 2, since the Haar measure $\hat{\mu}_4$ is $A$-ergodic, we may take an $A$-ergodic component $\mu'_4$ of $\mu_4$ projecting to the Haar measure with
\[
\lambda^{F}_{\text{top}}(s_1, \mu'_4) > 0.
\]
If $\lambda^{F}_{\mu'_4}, \ldots, \lambda^{F}_{\mu'_p} : A \to \mathbb{R}$ denote the fiberwise Lyapunov exponents for the $A$-invariant, $A$-ergodic measure $\mu'_4$ then $0 < \lambda^{F}_{\mu'^{1,3} \mu'_4}(s_1) = \lambda^{F}_{\text{top}}(s_1, \mu'_4)$ for some $1 \leq j'' \leq p''$ whence some fiberwise Lyapunov exponent $\lambda^{F}_{\mu'^{1,3} \mu'_4} : A \to \mathbb{R}$ is a non-zero linear functional.

This completes the proof of Proposition 12.2. \qed

13.4. **Modifications for general case.** When $\Gamma$ is a lattice in $\text{SL}(n, \mathbb{R})$ we replace the first averaging step with a slightly more complicated averaging.

**First averaging.** We again take $\mu_0 = \mu'$ to be the measure guaranteed by Proposition 12.1 with non-zero fiber exponent
\[
\lambda^{F}_{\mu'^{1,3}} : A \to \mathbb{R}, \quad \lambda^{F}_{\mu'^{1,3} \mu_0} \neq 0.
\]
Without loss of generality (by conjugating by a permutation matrix) we may assume that for the element
\[
s = \text{diag}(\frac{1}{2}, \ldots, 2, \ldots, 2)
\]
of $A \subset \text{SL}(n, \mathbb{R})$, we have
\[
\lambda^{F}_{\mu'^{1,3} \mu_0}(s) \neq 0.
\]
Take $s_0$ to be either $s$ or $s^{-1}$ so that $\lambda^{F}_{\mu'^{1,3} \mu_0}(s_0) > 0$. 

Consider the unipotent subgroup $U \subset \text{SL}(n, \mathbb{R})$ of matrices of the form

$$U = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & * & \cdots & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & * \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$ 

Note that $U$ commutes with $s_0$.

Let $F_n$ be a Følner sequence in $U$ and let $\mu_1$ be any weak-$*$ limit point of $\{F_n \ast \mu : T \geq 0\}$ where

$$F_n \ast \mu = \frac{1}{|F_n|} \int_{F_n} u_+ \mu \, du.$$

From fact analogous to those in Claim 13.1, $\mu_1$ is $s$-invariant and $\lambda_{\text{top}}^F(s, \mu_1) \geq \lambda_{\text{top}}^F(s, \mu_0)$.

We again average $\mu_1$ over a Følner sequence in $A$ and let $\mu_2$ be any weak-$*$ limit point of $\{A^T \ast \mu_1 : T \geq 0\}$. Then $\mu_2$ is $A$-invariant and

$$\lambda_{\text{top}}^F(s_0, \mu_2) \geq \lambda_{\text{top}}^F(s_0, \mu_1) > 0.$$

Again, we gave equality of the projected measures $\hat{\mu}_1 = \hat{\mu}_2$ so $\hat{\mu}_2$ is $U$-invariant and $A$-invariant. From Claim 13.2(4), $\hat{\mu}_2$ is invariant under the group

$$H = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & * & * & \cdots & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & * & \cdots & * & * \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$ 

As $\hat{\mu}_2$ is $A$-ergodic, we may replace $\mu_2$ with an $A$-ergodic component $\mu_2'$ of $\mu_2$ such that

1. $\lambda_{\text{top}}^F(s_0, \mu_2') > 0$, and
2. the projection of $\mu_2'$ to $\text{SL}(n, \mathbb{R})/\Gamma$ is $\hat{\mu}_2$.

Then, if $\lambda_{1,\mu_2'}, \ldots, \lambda_{p',\mu_2'} : A \to \mathbb{R}$ denote the fiberwise Lyapunov exponents for $\mu_2'$, we have $0 < \lambda_{j',\mu_2'}(s_0) = \lambda_{\text{top}}^F(s_0, \mu_2')$ for some $1 \leq j' \leq p'$.

**Second averaging.** Consider now the roots $\beta^{1,2}$ and $\beta^{1,n}$ of $G$. Since $\beta^{1,2}$ and $\beta^{1,n}$ are not proportional, at most one of $\beta^{1,2}$ and $\beta^{1,n}$ is proportional to $\lambda_{j',\mu_2'}$. In particular, we may find either $s$ or $\overline{s}$ in $A$ such that

1. $\beta^{1,2}(s) = 0$ but $\lambda_{j',\mu_2'}(s) \neq 0$; or
2. $\beta^{1,n}(\overline{s}) = 0$ but $\lambda_{j',\mu_2'}(\overline{s}) \neq 0$.

The two cases in the second averaging step of in Section 13.3 are identical to before, where we either average over the 1-parameter group $U^{1,2}$ in the case $\lambda_{j',\mu_2'}(s) \neq 0$ or $U^{1,n}$ in the case $\lambda_{j',\mu_2'}(\overline{s}) \neq 0$. The structure theory of $\text{SL}(n, \mathbb{R})$ will then imply that the measure obtained after the second averaging projects to the Haar measure.

**References**


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