LYAPUNOV EXPONENTS
AND INVARIANT CONFORMAL STRUCTURES,
WITH APPLICATIONS TO RIGIDITY PROBLEMS IN GEOMETRY

– PART I –

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Abstract. In the first part of the minicourse we will concentrate on properties of linear cocycles. A linear cocycle over a diffeomorphism $f$ of a manifold $M$ is an automorphism of a vector bundle over $M$ that projects to $f$. Important examples are the differential of $f$ and its restrictions to invariant sub-bundles of $TM$. We will discuss the notions of conformality and uniform quasiconformality of the cocycle, and the related notion of an invariant conformal structure. We will prove existence and continuity of an invariant conformal structure for a uniformly quasiconformal cocycle $A$ and give applications to rigidity of Anosov systems. Further, we will obtain conformality of a cocycle $A$ over a hyperbolic diffeomorphism $f$ from its periodic data, i.e. the values of $A$ at the periodic points of $f$. An important role in these arguments is played by fiber bunching and holonomies of the cocycle. Finally, we will consider cocycles with one Lyapunov exponent over hyperbolic and partially hyperbolic systems and obtain a structural theorem for them, which can be viewed as a continuous version of the Amenable Reduction.
1. Lecture 1

1.1. Linear cocycles.

Let $f$ be a diffeomorphism of a compact connected manifold $\mathcal{M}$ and let $P : \mathcal{E} \to \mathcal{M}$ be a finite dimensional vector bundle over $\mathcal{M}$. A linear cocycle over $f$ is an automorphism $A$ of $\mathcal{E}$ which projects to $f$, that is satisfies $P \circ A = f \circ P$.

We denote by $A_x : \mathcal{E}_x \to \mathcal{E}_{fx}$ the linear map between the fibers and use the following notations for the iterates of $A$: $A_x^0 = \text{Id}$, and for $n \in \mathbb{N}$.

$$A_x^n = A(f^{n-1}x) \circ \cdots \circ A(x) : \mathcal{E}_x \to \mathcal{E}_{f^n x} \quad \text{and} \quad A_x^{-n} = (A_{f^{-n}x})^{-1} : \mathcal{E}_x \to \mathcal{E}_{f^{-n}x}.$$

Clearly, $A$ satisfies the cocycle equation $A_{x+k}^n = A_x^n \circ A_k^n$.

In the case of a trivial vector bundle $\mathcal{E} = \mathcal{M} \times \mathbb{R}^d$, any linear cocycle $A$ can be identified with a matrix-valued function $A : \mathcal{M} \to GL(d, \mathbb{R})$ given by $A(x) = A_x$. We call such $A$ a $GL(d, \mathbb{R})$-valued cocycle.

For us, the primary example of a linear cocycle is the differential $Df$ viewed as an automorphism of the tangent bundle $T\mathcal{M}$ or its restriction to a $Df$-invariant sub-bundle $\mathcal{E}' \subset T\mathcal{M}$, such as the stable or unstable sub-bundle of a hyperbolic or partially hyperbolic system. In these examples,

$$A_x = D_x f \quad \text{or} \quad A_x = Df|_{\mathcal{E}'(x)} \quad \text{and} \quad A_x^n = D_x f^n \quad \text{or} \quad A_x^n = Df^n|_{\mathcal{E}'(x)}.$$

Since the stable and unstable sub-bundles are Hölder continuous but usually not more regular, we will assume that the fiber bundle $\mathcal{E}$ is Hölder continuous and we will consider linear cocycles in the Hölder category.

We fix a Hölder continuous Riemannian metric on $\mathcal{E}$. A linear cocycle $A$ is called Hölder continuous if the fiber maps $A_x$ depend Hölder continuously on $x$, i.e. there exist $K, \beta > 0$ such that for all nearby $x, y \in \mathcal{M}$

$$d(A_x, A_y) = \|A_x - A_y\| + \|(A_x)^{-1} - (A_y)^{-1}\| \leq K \cdot \text{dist}(x, y)^\beta$$

where $A_x$ and $A_y$ are viewed as matrices using local coordinates and $\|\cdot\|$ is the operator norm. In fact, the second term in the sum is not needed for continuous $A$ and compact $\mathcal{M}$ as $(A_x)^{-1}$ is then automatically continuous in $x$ and bounded on $\mathcal{M}$, so we can estimate

$$\|(A_x)^{-1} - (A_y)^{-1}\| = \|(A_x)^{-1} (A_y - A_x) (A_y)^{-1}\| \leq K' \cdot \|A_x - A_y\|.$$
1.2. Conformal structures.

A conformal structure on $\mathbb{R}^d$, $d \geq 2$, is a class of proportional inner products. The space $C^d$ of conformal structures on $\mathbb{R}^d$ can be identified with the space of real symmetric positive definite $d \times d$ matrices with determinant 1, which is isomorphic to $SL(d, \mathbb{R})/SO(d, \mathbb{R})$. The group $GL(d, \mathbb{R})$ acts transitively on $C^d$ via

$$X(C) = (\det X^T X)^{-1/d} \cdot X^T C X, \quad C \in C^d, \ X \in GL(d, \mathbb{R}).$$

The space $C^d$ carries a $GL(d, \mathbb{R})$-invariant Riemannian metric of non-positive curvature. The distance to the identity in this metric is

$$\text{dist}(\text{Id}, C) = \sqrt{d/2} \cdot (\log \lambda_1)^2 + \cdots + (\log \lambda_d)^2)^{1/2},$$

where $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of $C$.

If $C \in C^d$ and $X \in GL(d, \mathbb{R})$ is sufficiently close to the identity, then

$$\text{dist} (C, X(C)) \leq k(C) \cdot \|X - \text{Id} \|,$$

where $k(C)$ is bounded on compact sets in $C^d$ [KS10].

For a vector bundle $\mathcal{E} \to \mathcal{M}$ we can consider a bundle $\mathcal{C}$ over $\mathcal{M}$ whose fiber $\mathcal{C}_x$ is the space of conformal structures on $\mathcal{E}_x$. Using a background Riemannian metric on $\mathcal{E}$, the space $\mathcal{C}_x$ can be identified with the space of symmetric positive linear operators on $\mathcal{E}_x$ with determinant 1. We equip the fibers of $\mathcal{C}$ with the Riemannian metric as above. A continuous (measurable) section $\sigma$ of $\mathcal{C}$ is called a continuous (measurable) conformal structure on $\mathcal{E}$. A measurable conformal structure $\sigma$ is called bounded if the distance between $\sigma(x)$ and $\tau(x)$ is bounded on $\mathcal{M}$ for a continuous conformal structure $\tau$ on $\mathcal{M}$.

An invertible linear map $A : \mathcal{E}_x \to \mathcal{E}_y$ induces an isometry from $\mathcal{C}_x$ to $\mathcal{C}_y$ via

$$A(C) = (\det(A^* A))^{1/d} \cdot (A^{-1})^* C A^{-1},$$

where $C \in \mathcal{C}_x$ is a conformal structure viewed as an operator, and $A^*$ is the adjoint of $A$. If $A : \mathcal{E} \to \mathcal{E}$ is a linear cocycle over $f$, we say that a conformal structure $\sigma$ on $\mathcal{E}$ is $A$-invariant if

$$A_x(\sigma(x)) = \sigma(fx) \quad \text{for all } x \in \mathcal{M}. $$
1.3. Conformality and uniform quasiconformality.

The quasiconformal distortion is a measure of non-conformality of the cocycle.

**Definition 1.1.** The quasiconformal distortion of a cocycle $\mathcal{A}$ on $\mathcal{E}$ is the function

$$Q_{\mathcal{A}}(x,n) = \|A_x^n\| \cdot \| (A_x^n)^{-1} \| = \max \{ \| A_x^n(v) \| : v \in \mathcal{E}_x, \| v \| = 1 \} \cdot \min \{ \| A_x^n(v) \| : v \in \mathcal{E}_x, \| v \| = 1 \}, \quad x \in X \text{ and } n \in \mathbb{Z}.$$

If $Q_{\mathcal{A}}(x,n) \leq K$ for all $x$ and $n$, the cocycle is called uniformly quasiconformal, and if $Q_{\mathcal{A}}(x,n) = 1$ for all $x$ and $n$, it is called conformal.

Clearly, a cocycle $\mathcal{A}$ is conformal with respect to a Riemannian metric on $\mathcal{E}$ if and only if it preserves the conformal structure associated with this metric. We note that the notion of uniform quasiconformality does not depend on the choice of a continuous metric on $\mathcal{E}$. So if $\mathcal{A}$ preserves a continuous conformal structure on $\mathcal{E}$ then $\mathcal{A}$ is uniformly quasiconformal on $\mathcal{E}$ with respect to any continuous metric on $\mathcal{E}$. Theorem 1.3 below shows that the converse is also true if $f$ is a transitive Anosov diffeomorphism.

In the next proposition we apply observations made by D. Sullivan and P. Tukia for quasiconformal group actions to our case. Note that the converse statement is also true.

**Proposition 1.2.** Let $f$ be a diffeomorphism of a compact manifold $\mathcal{M}$ and let $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ be a continuous linear cocycle over $f$. If $\mathcal{A}$ is uniformly quasiconformal then it preserves a bounded Borel measurable conformal structure $\sigma$ on $\mathcal{E}$.

**Proof.** Let $\tau$ be a continuous conformal structure on $\mathcal{E}$. We denote by $\tau(x)$ the conformal structure on $\mathcal{E}_x$, $x \in \mathcal{M}$. We consider the set

$$S(x) = \{ (A^{-n}_x)(\tau(f^n)) : n \in \mathbb{Z} \}$$

in $\mathcal{C}(x)$, the space of conformal structures on $\mathcal{E}_x$. Since $\mathcal{A}$ is uniformly quasiconformal, the sets $S(x)$ have uniformly bounded diameters. Since the space $\mathcal{C}_x$ has non-positive curvature, for every $x$ there exists a uniquely determined closed ball of the smallest radius containing $S(x)$. We denote its center by $\sigma(x)$.

It follows from the construction that the conformal structure $\sigma$ is $\mathcal{A}$-invariant and its distance from $\tau$ is bounded. For any $k \geq 0$ the set

$$S_k(x) = \{ (A^{-n}_x)(\tau(f^n)) : \| n \| \leq k \}$$

depends continuously on $x$ in Hausdorff distance, and so does the center $\sigma_k(x)$ of the smallest ball containing $S_k(x)$. Since $S_k(x) \to S(x)$ as $k \to \infty$ for any $x$, the conformal structure $\sigma$ is the pointwise limit of continuous conformal structures $\sigma_k(x)$. Hence $\sigma$ is Borel measurable. $\square$

**Theorem 1.3.** [S02, KS10] Let $f$ be a transitive Anosov diffeomorphism and let $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ be a Hölder continuous linear cocycle over $f$. If $\mathcal{A}$ is uniformly quasiconformal then it preserves a Hölder continuous conformal structure on $\mathcal{E}$, equivalently, $\mathcal{A}$ is conformal with respect to a Hölder continuous Riemannian metric on $\mathcal{E}$.

Proof of Hölder continuity will be given in the next lecture.
1.4. An application: smoothness of stable/unstable sub-bundles and rigidity for uniformly quasiconformal Anosov diffeomorphisms. (Definition in Sec. 2.1.)

**Theorem 1.4 ([S02]).** Let \( f \) be a transitive \( C^\infty \) Anosov diffeomorphism of \( \mathcal{M} \). Suppose that \( f \) is uniformly quasiconformal on \( E^u \) with \( \dim E^u \geq 2 \). Then

(i) \( f \) is conformal with respect to a Riemannian metric on \( E^u \) which is Hölder continuous on \( \mathcal{M} \) and \( C^\infty \) along the leaves of the unstable foliation,

(ii) The holonomy maps of the stable foliation are conformal, and \( E^s \) is \( C^\infty \).

**Proof.** (i) By the previous theorem, \( \mathcal{A} \) preserves a Hölder continuous conformal structure on \( \mathcal{E} \). We show its smoothness along the leaves using non-stationary linearization.

**Proposition 1.5. (Non-stationary linearization for \( \frac{1}{2} \) pinched contractions) [S02]**

Let \( f \) be a diffeomorphism of a compact Riemannian manifold \( \mathcal{M} \), and let \( W \) be a continuous invariant foliation with \( C^\infty \) leaves. Suppose that \( \|Df|_{T_E^u}\| < 1 \), and there exist \( K > 0 \) and \( \theta < 1 \) such that for all \( x \in \mathcal{M} \) and \( n \in \mathbb{N} \),

\[
(1.3) \quad \| (Df^n|_{T_{E^u}^u})^{-1} \| \cdot \| Df^n|_{T_{E^u}^u} \|^2 \leq K \theta^n.
\]

Then for every \( x \in \mathcal{M} \) there exists a \( C^\infty \) diffeomorphism \( h_x : W(x) \to T_xW \) such that

(i) \( h_f \circ f = D_x f \circ h_x \),

(ii) \( h_x(x) = 0 \) and \( D_x h_x \) is the identity map,

(iii) \( h_x \) depends continuously on \( x \) in \( C^\infty \) topology.

Suppose \( f \) preserves a continuous conformal structure \( \tau \) on \( E^u \). Then (1.3) is satisfied for \( f^{-1} \) and we get a non-stationary linearization \( h_x \) along \( W^u \). For each \( x \in \mathcal{M} \) we extend \( \tau(x) \) to a constant conformal structure \( \sigma \) on \( T E^u_x \): for any \( t \in E^u_x \) we denote by \( \sigma(t) \) the push forward of \( \tau(x) \) by translation from 0 to \( t \).

**Proposition 1.6 ([S02]).** Each \( h_x \) is conformal, i.e. it pushes \( \tau \) on \( W^u(x) \) to \( \sigma \) on \( E^u(x) \) and hence \( \tau \) is \( C^\infty \) along the leaves of \( W^u \).

**Proof.** We need to show that for any \( y \in W^u(x) \), \( h_x(\tau(y)) = \sigma(h_x(y)) \). We iterate by \( f^{-1} \).

By continuity of \( \tau \) and properties (ii,iii) of \( h_x \), for each \( \epsilon > 0 \) there exists \( n > 0 \) such that

\[
\text{dist}(h_{f^{-n}x}(\tau(\epsilon y)), \sigma(h_{f^{-n}x}(\epsilon y))) < \epsilon.
\]

To obtain the following equalities, we note that \( Df^n \) induces an isometry between the spaces of conformal structures, \( \tau \) is \( f \)-invariant, \( \sigma \) is \( Df \)-invariant, and \( h_x(y) = Df^n(h_{f^{-n}x}(\epsilon y)) \) by Proposition 1.5 (i).

Thus,

\[
\begin{align*}
\epsilon &> \text{dist}(h_{f^{-n}x}(\tau(\epsilon y)), \sigma(h_{f^{-n}x}(\epsilon y))) \\
&= \text{dist}(Df^n(h_{f^{-n}x}(\tau(\epsilon y))), Df^n(\sigma(h_{f^{-n}x}(\epsilon y)))) \\
&= \text{dist}(Df^n(h_{f^{-n}x}(\epsilon y)), \sigma(Df^n(h_{f^{-n}x}(\epsilon y)))) \\
&= \text{dist}(h_x(\tau(y)), \sigma(h_x(y))).
\end{align*}
\]

As the above holds for any \( \epsilon > 0 \), it follows that \( h_x(\tau(y)) = \sigma(h_x(y)) \). \qed
(ii) Let $x$ and $y$ be two nearby points in $\mathcal{M}$. We consider the holonomy map of the stable foliation $\mathcal{H}_{x,y} : W^u_{\text{loc}}(x) \to W^u_{\text{loc}}(y)$:

$$z \in W^u_{\text{loc}}(x) \mapsto \mathcal{H}_{x,y}(z) = W^u_{\text{loc}}(y) \cap W^s_{\text{loc}}(z).$$

First we show that the holonomy maps $\mathcal{H}_{x,y}$ are conformal, i.e. $\mathcal{H}_{x,y}(\tau(z)) = \tau(\mathcal{H}_{x,y}(z))$. By the $C^r$-section Theorem [HPS], pinching given by conformality of $f$ on $E^u$ implies that the stable distribution is $C^1$, and hence the holonomy maps are uniformly $C^1$ [PSW]. It suffices to consider the case when $z = x$ and $y \in W^s(x)$ so that $y = \mathcal{H}_{x,y}(z)$. We iterate by $f$ and note that $\mathcal{H}_{x,y} = f^{-n} \circ \mathcal{H}_{f^n x, f^n y} \circ f^n$ and that $D f^n_x \mathcal{H}_{f^n x, f^n y}$ is close to identity for large $n$ since $f^n y$ is close to $f^n x$. Since $\tau$ is continuous, $\tau(f^n x)$ is close to $\tau(f^n y)$. Thus, $\mathcal{H}_{f^n x, f^n y}(\tau(f^n x))$ is close to $\tau(f^n y)$. Since $f^{-n}$ induces an isometry between the spaces of conformal structures on $E^u(f^n y)$ and on $E^u(y)$, we conclude that $\mathcal{H}_{x,y}(\tau(x))$ is close to $\tau(y)$, so by letting $n \to \infty$, we get $\mathcal{H}_{x,y}(\tau(z)) = \tau(\mathcal{H}_{x,y}(z))$.

Now we show that $\mathcal{H}_{x,y}$ are $C^\infty$. Using the non-stationary linearization coordinates we view it as the map

$$G_{x,y} = h_y \circ \mathcal{H}_{x,y} \circ h_x^{-1} : E^u_x \to E^u_y,$$

which is a conformal $C^1$ diffeomorphism defined on a neighborhood of 0, and hence is $C^\infty$. Indeed, if $n > 2$, it is Möbius, and if $n = 2$ it is complex analytic.

It follows that $E^s$ is $C^\infty$ [PSW].

\begin{theorem}[[KS03]]. (Global rigidity of UQC Anosov diffeomorphisms) \end{theorem}

Let $f$ be a transitive $C^\infty$ Anosov diffeomorphism of $\mathcal{M}$ which is uniformly quasiconformal on the stable and unstable sub-bundles. Suppose either that both sub-bundles have dimension at least three, or that they have dimension at least two and $\mathcal{M}$ is an infranilmanifold. Then $f$ is $C^\infty$ conjugate to an affine Anosov automorphism of a finite factor of a torus.

\begin{proof}

Ideas of the proof. Theorem 1.4 implies that both $E^u$ and $E^s$ are $C^\infty$ and there is a $C^\infty$ Riemannian metric on $\mathcal{M}$ for which $f$ is conformal on both $E^u$ and $E^s$. In general, smoothness of both $E^u$ and $E^s$ is not known to imply smooth conjugacy to algebraic model (although this is conjectured). Results of Benoist and Labourie [BL93] give such a conjugacy if $f$ preserves a smooth affine connection.

Under the assumptions of the theorem, we showed that the holonomy maps $\mathcal{H}_{x,y}$ are globally defined affine conformal maps between the leaves and then that the non-stationary linearization coordinates $h_x^s$ and $h_x^u$ depend $C^\infty$ on $x$. This allowed us to construct a smooth invariant affine connection.

\end{proof}
2. Lecture 2

2.1. Anosov and partially hyperbolic diffeomorphisms - definitions.

A diffeomorphism $f$ of a compact manifold $M$ is called **partially hyperbolic** if there exist a nontrivial $Df$-invariant splitting of the tangent bundle $T\mathcal{M} = E^s \oplus E^c \oplus E^u$, and a Riemannian metric on $\mathcal{M}$ for which one can choose continuous functions $\nu < 1$, $\hat{\nu} < 1$, $\gamma$, $\hat{\gamma}$ such that for all $x \in \mathcal{M}$ and unit vectors $v^s \in E^s(x)$, $v^c \in E^c(x)$, and $v^u \in E^u(x)$

\begin{equation}
\|Df(v^s)\| < \nu(x) < \gamma(x) < \|Df(v^c)\| < \hat{\gamma}(x)^{-1} < \hat{\nu}(x)^{-1} < \|Df(v^u)\|.
\end{equation}

The sub-bundles $E^s$, $E^u$, and $E^c$ are called, respectively, stable, unstable, and center. $E^s$ and $E^u$ are tangent to the stable and unstable foliations $W^s$ and $W^u$ respectively. If $E^c$ is trivial, $f$ is called **hyperbolic** or **Anosov**. Most of the results for Anosov case below also have direct analogs for more general hyperbolic systems such as locally maximal hyperbolic sets and subshifts of finite type.

We define the **local stable manifold** of $x$, $W^s_{\text{loc}}(x)$, as a ball centered at $x$ of radius $\rho$ in the intrinsic metric of $W^s(x)$. We choose $\rho$ sufficiently small so that for every $x \in \mathcal{M}$ we have $\|Df_y\| < \nu(x)$ for all $y$ in $W^s_{\text{loc}}(x)$. Local unstable manifolds are defined similarly.

For hyperbolic $f$ we can choose $\rho$ so that $W^s_{\text{loc}}(x) \cap W^u_{\text{loc}}(z)$ consists of a single point for any sufficiently close $x$ and $z$ in $\mathcal{M}$. This property is called the local product structure of the stable and unstable foliations. We say that a measure $\mu$ has **local product structure** if it is locally equivalent to the product of measures on $W^s_{\text{loc}}(x)$ and $W^u_{\text{loc}}(x)$

For partially hyperbolic $f$, an **su-path** in $\mathcal{M}$ is defined as a concatenation of finitely many subpaths which lie entirely in a single leaf of $W^s$ or $W^u$ and $f$ is called **accessible** if any two points in $\mathcal{M}$ can be connected by an su-path.

The diffeomorphism $f$ is called **center bunched** if the functions $\nu, \hat{\nu}, \gamma, \hat{\gamma}$ can be chosen to satisfy

$$\nu < \gamma \hat{\gamma} \quad \text{and} \quad \hat{\nu} < \gamma \hat{\gamma}.$$

It follows that $\|Df|_{E^s}\| \cdot \|(Df|_{E^c})^{-1}\| \cdot \nu < \gamma^{-1} \gamma^{-1} \nu < 1$ and similarly with $\hat{\nu}$.

We say that $f$ is **volume-preserving** if it has an invariant probability measure $\mu$ in the measure class of a smooth volume. If $f$ is essentially accessible, $C^2$ and center bunched then $\mu$ is ergodic [BW10].
2.2. Fiber bunching and holonomies. Next we define fiber bunching of a cocycle. This condition means that non-conformality of the cocycle is dominated, in a sense, by the contraction and expansion in the base given by the functions \( \nu \) and \( \hat{\nu} \) in (2.1). In particular, bounded, conformal, and uniformly quasiconformal cocycles are fiber bunched. We use the weakest, “pointwise”, version of fiber bunching with non-constant estimates of expansion and contraction in the base.

**Definition 2.1.** A \( \beta \)-Hölder cocycle \( \mathcal{A} \) over a (partially) hyperbolic diffeomorphism \( f \) is called fiber bunched if there exist \( \theta < 1 \) and \( L \) such that for all \( x \in \mathcal{M} \) and \( n \in \mathbb{N} \),

\[
Q_{\mathcal{A}}(x, n) \cdot (\nu_{x}^{n})^{\beta} = \| \mathcal{A}_{x}^{n} \| \cdot (\nu_{x}^{n})^{\beta} < L \theta^{n} \quad \text{and} \quad Q_{\mathcal{A}}(x, -n) \cdot (\hat{\nu}_{x}^{-n})^{\beta} < L \theta^{n},
\]

where \( \nu_{x}^{n} = \nu(f^{n-1}x) \cdots \nu(x) \) and \( \hat{\nu}_{x}^{-n} = (\hat{\nu}(f^{-n}x))^{-1} \cdots (\hat{\nu}(f^{-1}x))^{-1} \).

This condition plays an important role in the study of non-commutative cocycles and in particular ensures existence of holonomies, which we define next. This terminology for linear cocycles was introduced in [V08] and the holonomies were further studied in [ASV13, KS13, S15]. The result below gives existence of holonomies under the weakest fiber bunching assumption (2.2).

**Proposition 2.2. (Existence of holonomies)** Suppose that \( \mathcal{A} \) is a \( \beta \)-Hölder fiber bunched linear cocycle over \( f \). Then for every \( x \in \mathcal{M} \) and \( y \in W^{s}(x) \) the limit

\[
H_{x,y}^{A,s} = \lim_{n \to \infty} (\mathcal{A}_{y}^{n})^{-1} \circ \mathcal{A}_{x}^{n},
\]

exists and satisfies

\begin{enumerate}
  \item[(H1)] \( H_{x,y}^{A,s} \) is a linear map from \( \mathcal{E}_{x} \) to \( \mathcal{E}_{y} \) which depends continuously on \( (x, y) \);
  \item[(H2)] \( H_{x,x}^{A,s} = \text{Id} \) and \( H_{y,z}^{A,s} \circ H_{x,y}^{A,s} = H_{x,z}^{A,s} \), which implies \( (H_{x,y}^{A,s})^{-1} = H_{y,x}^{A,s} \);
  \item[(H3)] \( H_{x,y}^{A,s} = (\mathcal{A}_{y}^{n})^{-1} \circ H_{x,y}^{f^{n}x, f^{n}y} \circ \mathcal{A}_{x}^{n} \) for all \( n \in \mathbb{N} \);
  \item[(H4)] \( \| H_{x,y}^{A,s} - \text{Id} \| \leq c \text{dist}(x, y)^{\beta} \), where \( c \) is independent of \( x \) and \( y \in W^{s}_{\text{loc}}(x) \).
\end{enumerate}

The family of maps \( H_{x,y}^{A,s}, \ x \in \mathcal{M}, \ y \in W^{s}(x), \) defined by (2.3) is called the (standard) stable holonomy for \( \mathcal{A} \). We note that for a \( \beta \)-Hölder fiber bunched cocycle (2.3) is the only family of maps satisfying (H1,2,3,4) [KS13], but there may be other families satisfying (H1,2,3) and (H4) with exponent \( \beta' < \beta \) [KS16].

The unstable holonomy \( H_{x,y}^{A,u} \) is defined similarly:

\[
H_{x,y}^{A,u} = \lim_{n \to \infty} ((\mathcal{A}_{y}^{n})^{-1} \circ (\mathcal{A}_{x}^{n})) = \lim_{n \to \infty} (\mathcal{A}_{f^{-n}y}^{n} \circ (\mathcal{A}_{f^{-n}x}^{n})^{-1}), \quad \text{where} \ y \in W^{u}(x).
\]

It satisfies (H1, 2, 4) and

\[
(H3') \ H_{x,y}^{A,u} = (\mathcal{A}_{y}^{-n})^{-1} \circ H_{f^{-n}x, f^{-n}y}^{A,u} \circ \mathcal{A}_{x}^{-n} \quad \text{for all} \ n \in \mathbb{N}.
\]
2.3. Continuity of a measurable invariant conformal structure.

Theorem 1.3 follows from Proposition 1.2 and the next theorem since UQC implies fiber bunching and we can take $\mu$ to be, for example, the measure of maximal entropy, which satisfies the assumptions on $\mu$ in Theorem 2.3.

**Theorem 2.3.** [KS13] Let $f$ be a transitive Anosov diffeomorphism, $A : E \to E$ be a $\beta$-Hölder fiber bunched linear cocycle over $f$, and $\mu$ be an ergodic $f$-invariant probability measure with full support and local product structure. Then any $A$-invariant $\mu$-measurable conformal structure on $E$ is $\beta$-Hölder.

We also have a similar result for volume preserving partially hyperbolic case. Note that only continuity is obtained in this case.

**Theorem 2.4.** [KS13] Let $f$ be a center bunched accessible partially hyperbolic diffeomorphism, $A : E \to E$ be a $\beta$-Hölder fiber bunched cocycle over $f$, and $\mu$ be an $f$-invariant volume. Then any $A$-invariant $\mu$-measurable conformal structure $\tau$ on $E$ is continuous.

Let $\tau$ be an $A$-invariant $\mu$-measurable conformal structure on $E$. Both results follow from the next proposition which shows that $\tau$ is essentially invariant under the stable and unstable holonomies of $A$. It applies to both hyperbolic and partially hyperbolic case since it uses only contraction of $W^s$ and ergodicity of $\mu$.

**Proposition 2.5.** Let $H^s$ be the stable holonomy for a linear cocycle $A$. If $\tau$ is a $\mu$-measurable $A$-invariant conformal structure then $\tau$ is essentially $H^s$-invariant, i.e. there is a set $G \subset M$ of full measure such that

$$\tau(y) = H^s_{xy}(\tau(x)) \text{ for all } x, y \in G \text{ such that } y \in W^s_{\text{loc}}(x).$$

**Proof.** We denote by $A^n_x(\sigma)$ the push forward of a conformal structure $\sigma$ from $E_x$ to $E_{f^n x}$ induced by $A^n_x$, and similar notations for push forwards by $H^s$. We also let $x_i = f^i x$. Since $A^n$ induces an isometry and $\tau$ is $A$-invariant, we obtain

$$\text{dist} (\tau(y), H^s_{xy}(\tau(x))) = \text{dist} (A^n_y(\tau(y)), A^n_{xy}(\tau(x))) =$$

$$= \text{dist}(\tau(y_n), H^s_{x_n y_n}(\tau(x))) = \text{dist}(\tau(y_n), H^s_{x_n y_n}(\tau(x))) \leq$$

$$\leq \text{dist}(\tau(y_n), \tau(x_n)) + \text{dist}(\tau(x_n), H^s_{x_n y_n}(\tau(x_n))).$$

Since $\tau$ is $\mu$-measurable, by Lusin’s Theorem there exists a compact set $S \subset M$ with $\mu(S) > 1/2$ on which $\tau$ is uniformly continuous and hence bounded. Let $G$ be the set of points in $M$ for which the frequency of visiting $S$ equals $\mu(S) > 1/2$. By Birkhoff Ergodic Theorem, $\mu(G) = 1$.

Suppose that both $x$ and $y$ are in $G$. Then there exists a sequence $\{n_i\}$ such that $x_{n_i} \in S$ and $y_{n_i} \in S$. Since $y \in W^s_{\text{loc}}(x)$, $\text{dist}(x_{n_i}, y_{n_i}) \to 0$ and hence $\text{dist}(\tau(x_{n_i}), \tau(y_{n_i})) \to 0$ by uniform continuity of $\tau$ on $S$. Since $H^s$ is continuous and satisfies $H^s_{xx} = \text{Id}$, we have $\|H^s_{x_{n_i} y_{n_i}} - \text{Id}\| \to 0$. Since $\tau$ is bounded on $S$,

$$\text{dist}(\tau(x_{n_i}), H^s_{x_{n_i} y_{n_i}}(\tau(x_{n_i}))) \to 0.$$

We conclude that $\text{dist} (\tau(y), H^s_{xy}(\tau(x))) = 0$ and thus $\tau$ is essentially $H^s$-invariant. $\square$
Similarly, $\tau$ is essentially $H^u$-invariant. By Hölder continuity of holonomies (H4) this implies essential Hölder continuity of $\tau$ along stable and unstable leaves:

$$\dist(\tau(x),\tau(y)) \leq C \dist(x,y)^B \text{ if } y \in W_{\text{loc}}^{s/u}(x) \text{ and both } x, y \in G.$$ 

To complete the proof of Theorem 2.3 we use the local product structure of $\mu$ and the local product structure of the stable and unstable foliations to show that $\tau$ coincides $\mu$-a.e. with a Hölder continuous conformal structure on $\text{supp } \mu = \mathcal{M}$.

In the proof of Theorem 2.4 the local product structure is replaced by accessibility. In this case the global continuity of $\tau$ on $\mathcal{M}$ follows from [ASV13, Theorem E] or [W13, Theorem 4.2]. If one makes a stronger accessibility assumption, a simpler argument yields Hölder continuity of $\tau$. \hfill \square

2.4. Obtaining fiber bunching and conformality from periodic data.

Abundance of periodic points is one of the key features of hyperbolic systems. We consider the periodic data of $\mathcal{A}$, i.e. $\{\mathcal{A}_p^n : p = f^n p\}$.

In the previous results we needed fiber bunching to get the holonomies. The next proposition allows us to obtain fiber bunching of a cocycle $\mathcal{A}$ from fiber bunching of its periodic data.

Proposition 2.6. [KS10, S17] Let $\mathcal{A}$ be a $\beta$-Hölder cocycle over an Anosov diffeomorphism $f : \mathcal{M} \to \mathcal{M}$. Suppose that there exist $\tilde{\theta} < 1$ and $\tilde{L}$ such that whenever $f^{n}p = p$,

\begin{align}
Q_{\mathcal{A}}(p,n) \cdot (\nu_p^n)^B < \tilde{L} \tilde{\theta}^n & \quad \text{and} \quad Q_{\mathcal{A}}(p,-n) \cdot (\hat{\nu}_p^{-n})^B < \tilde{L} \tilde{\theta}^n.
\end{align}

Then $\mathcal{A}$ is fiber bunched.

The proof uses results on subadditive sequences of continuous functions and on periodic approximation of Lyapunov exponents of measures [K11].

If a cocycle $\mathcal{A}$ over an Anosov diffeomorphism $f$ is conformal or uniformly quasiconformal then, clearly, there exists a constant $C_{\text{per}}$ such that $Q_{\mathcal{A}}(p,n) \leq C_{\text{per}}$ for every periodic point $p$ and $n$ such that $p = f^n p$. The next theorem shows that the converse is also true. A similar result was also obtained in [LW10] under an extra bunching-type assumption.

Theorem 2.7. [KS10] Let $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ be a Hölder continuous linear cocycle over a transitive $\mathcal{C}^2$ Anosov diffeomorphism $f$. If there exists a constant $C_{\text{per}}$ such that

\begin{align}
Q_{\mathcal{A}}(p,n) = \|\mathcal{A}_p^n\| \cdot \|(\mathcal{A}_p^n)^{-1}\| \leq C_{\text{per}} \quad \text{whenever } f^n p = p,
\end{align}

then $\mathcal{A}$ is conformal with respect to a Hölder continuous Riemannian metric on $\mathcal{E}$.

Proof. First, the assumption (2.5) and Proposition 2.6 give that $\mathcal{A}$ is fiber bunched. By Theorem 1.3 it suffices to show that $\mathcal{A}$ is uniformly quasiconformal on $\mathcal{E}$. We do this using a dense orbit argument. Since $f$ is transitive, there exists a point $z \in \mathcal{M}$ with dense orbit $\mathcal{O} = \{f^k z : k \in \mathbb{Z}\}$. We will show that the quasiconformal distortion $Q_{\mathcal{A}}(z,k)$ is uniformly bounded in $k \in \mathbb{Z}$. Since $\mathcal{O}$ is dense and $Q_{\mathcal{A}}(x,k)$ is continuous on $\mathcal{M}$ for each $k$, this implies that $Q_{\mathcal{A}}(x,k)$ is uniformly bounded in $x \in X$ and $k \in \mathbb{Z}$. 

We consider any two points of $\mathcal{O}$ with $\text{dist}(f^{k_1}z, f^{k_2}z) < \delta_0$, where $\delta_0$ is sufficiently small to apply the Anosov Closing Lemma [KtH, Theorem 6.4.15]. We assume that $k_1 < k_2$ and denote $x = f^{k_1}z$ and $n = k_2 - k_1$, so that $\delta = \text{dist}(x, f^nx) < \delta_0$. By the Anosov Closing Lemma there exists $p \in \mathcal{M}$ with $f^np = p$ such that $\text{dist}(f^ix, f^ip) \leq c\delta$ for $i = 0, \ldots, n$.

Let $y$ be a point in $W^s_{\text{loc}}(p) \cap W^u_{\text{loc}}(x)$. Similarly to Proposition 2.2 (H4), fiber bunching implies
\[
\|(A^n_p)^{-1} \circ A^n_y - \text{Id}\| \leq C\delta^\beta \quad \text{and} \quad \|(A^{-n}_y)^{-1} \circ A^{-n}_x - \text{Id}\| \leq C\delta^\beta.
\]
It follows that if $\delta_0$ is sufficiently small,
\[
Q_A(y, n)/Q_A(p, n) \leq (1 + C\delta^\beta)/(1 - C\delta^\beta) \leq 2 \quad \text{and} \quad Q_A(x, n)/Q_A(y, n) \leq 2.
\]
Thus $Q_A(x, n) \leq 4Q_A(p, n) \leq 4C_{\text{per}}$.

We take $m > 0$ such that the set $\{f^jz; |j| \leq m\}$ is $\delta_0$-dense in $\mathcal{M}$. Let
\[
Q_m = \max\{Q_A(z, j); |j| \leq m\}.
\]
Then for any $k > m$ there exists $j$, $|j| \leq m$, such that $\text{dist}(f^kz, f^jz) \leq \delta_0$ and hence
\[
Q_A(z, k) \leq Q_A(z, j) \cdot Q_A(f^jz, k\text{-}j) \leq Q_m \cdot 4C_{\text{per}}.
\]
The case of $k < -m$ is considered similarly. Thus $Q_A(z, k)$ is uniformly bounded. $\square$

2.5. **Weaker assumptions on periodic data.**

One may try to make weaker assumptions on periodic data, for example that for each $p$, there is a uniform bound $C(p)$ on $Q_A(p, n)$ for all periods $n$. This is equivalent to each of the following three statements:

- $A^n_p$ is diagonalizable over $\mathbb{C}$ with its eigenvalues equal in modulus;
- $A^n_p$ is conjugate to a conformal linear map;
- There exists a $A^n_p$-invariant conformal structure on $\mathcal{E}_p$.

In fact, the periodic assumption in the first part of Theorem 2.7 is equivalent to having such conformal structures for all periodic points uniformly bounded. Our next result for two-dimensional bundles does not require any extra assumptions.

**Theorem 2.8.** [KS10] Let $A : \mathcal{E} \to \mathcal{E}$ be a Hölder continuous linear cocycle over a transitive $C^2$ Anosov diffeomorphism $f$. Suppose that the fibers of $\mathcal{E}$ are two-dimensional. If for each periodic point $p \in \mathcal{M}$, the return map $A^n_p : \mathcal{E}_p \to \mathcal{E}_p$ is diagonalizable over $\mathbb{C}$ and its eigenvalues are equal in modulus, then $A$ is conformal with respect to a Hölder continuous Riemannian metric on $\mathcal{E}$.

We note that while Theorem 2.7 holds in any dimension, Theorem 2.8 does not hold in dimension higher than 2, as the following example shows.
Proposition 2.9. [KS10] Let $f$ be an Anosov diffeomorphism of $\mathcal{M}$ and let $\mathcal{E} = \mathcal{M} \times \mathbb{R}^d$, $d \geq 3$. For any $\epsilon > 0$ there exists a Lipschitz continuous linear cocycle $A : \mathcal{E} \to \mathcal{E}$, which is $\epsilon$-close to the identity, such that for all periodic points $p \in \mathcal{M}$ the return maps $A^n_p : \mathcal{E}_p \to \mathcal{E}_p$ are conjugate to orthogonal maps, but $A$ is not conformal with respect to any continuous Riemannian metric on $\mathcal{E}$.

Outline of the construction. Let $\mathcal{E} = \mathcal{M} \times \mathbb{R}^3$ and $A_x = \begin{bmatrix} \cos \alpha(x) & -\sin \alpha(x) & \epsilon \\ \sin \alpha(x) & \cos \alpha(x) & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Let $S$ be a closed $f$-invariant set in $\mathcal{M}$ without periodic points and let $\alpha : \mathcal{M} \to \mathbb{R}$ be a Lipschitz continuous function satisfying $\alpha(x) = 0$ for $x \in S$ and $0 < \alpha(x) \leq \epsilon$ for $x \notin S$.

Then for $x \in S$, $A^n_p = \begin{bmatrix} 1 & 0 & n\epsilon \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and so $Q_A(x, n) \to \infty$ as $n \to \infty$. Thus $A$ cannot be conformal.

At $p = f^n p$, the map $A^n_p$ has eigenvalues of modulus 1 and is diagonalizable if the $2 \times 2$ rotation block has complex eigenvalues, that is the rotation angle $\alpha(p) + \cdots + \alpha(f^{n-1} p)$ does not equal $\pi k$. This can be ensured for all periodic points by slightly modifying the function $\alpha$, if necessary. \hfill \square

Finally, we note that Theorem 2.8 and 2.7 were motivated by the study of rigidity for conformal Anosov systems. In particular, Theorem 2.8 as well as some other results in this lecture, were obtained for the restrictions of $Df$ to an invariant sub-bundle of $TM$ for Anosov $f$ [KS09]. Theorem 2.8 and 2.7, together with rigidity results for conformal Anosov systems like Theorem 1.7, allow to obtain global rigidity from periodic data. More generally, they can be applied to $Df$-invariant sub-bundles inside $E^u$ and $E^s$. For example, Theorem 2.8 was used to obtain a local rigidity result for a broad class of irreducible Anosov automorphisms of $\mathbb{T}^d$ in [GKS11].
3. Lecture 3

3.1. Lyapunov exponents and their periodic approximation.

For any measure $\mu$ on $\mathcal{M}$, any vector bundle $\mathcal{E}$ over $\mathcal{M}$ is trivial on a set of full measure and hence any linear cocycle $\mathcal{A}$ can be viewed as a $GL(d, \mathbb{R})$-valued cocycle on a set of full measure. By Oseledets Multiplicative Ergodic Theorem, for any continuous linear cocycle $\mathcal{A}$ over $f$, the Lyapunov exponents of $\mathcal{A}$ and Lyapunov decomposition of $\mathcal{E}$ are defined almost everywhere for every ergodic $f$-invariant measure $\mu$. We note that the exponents and the decomposition depend on the choice of $\mu$.

**Theorem 3.1. (Oseledets Multiplicative Ergodic Theorem)** Let $f$ be an invertible ergodic measure-preserving transformation of a Lebesgue probability space $(X, \mu)$. Let $\mathcal{A}$ be a measurable $GL(d, \mathbb{R})$-valued cocycle over $f$ satisfying $\log \|\mathcal{A}_x\| \in L^1(X, \mu)$ and $\log \|\mathcal{A}_x^{-1}\| \in L^1(X, \mu)$.

Then there exist numbers $\lambda_1 < \cdots < \lambda_\ell$, an $f$-invariant set $\Lambda$ with $\mu(\Lambda) = 1$, and an $\mathcal{A}$-invariant Lyapunov decomposition

$$\mathcal{E}_x = \mathcal{E}_x^1 \oplus \cdots \oplus \mathcal{E}_x^\ell$$

for $x \in \Lambda$

such that

(i) $\lim_{n \to \pm \infty} n^{-1} \log \|\mathcal{A}_x^n v\| = \lambda_i$ for any $i = 1, \ldots, \ell$ and any $0 \neq v \in \mathcal{E}_x^i$, and

(ii) $\lim_{n \to \pm \infty} n^{-1} \log |\det \mathcal{A}_x^n| = \sum_{i=1}^{\ell} m_i \lambda_i$, where $m_i = \dim \mathcal{E}_x^i$.

The numbers $\lambda_1, \ldots, \lambda_\ell$ are called the *Lyapunov exponents* of $\mathcal{A}$ with respect to $\mu$.

By Lyapunov exponents of $\mathcal{A}$ at a periodic point $p$ we mean the Lyapunov exponents of $\mathcal{A}$ with respect to the invariant measure on the orbit of $p$. They equal $(1/n)$ of the logarithms of the absolute values of the eigenvalues of $\mathcal{A}_p^n$, where $n$ is a period of $p$.

**Theorem 3.2 (Periodic approximation of Lyapunov exponents).** [K11]

Let $(\mathcal{M}, f)$ be a hyperbolic dynamical system, let $\mathcal{A}$ be a Hölder continuous linear cocycle over $f$, and let $\mu$ be an ergodic invariant measure for $f$.

Then the Lyapunov exponents $\lambda_1 \leq \cdots \leq \lambda_d$ of $\mathcal{A}$ with respect to $\mu$, listed with multiplicities, can be approximated by the Lyapunov exponents of $\mathcal{A}$ at periodic points. More precisely, for any $\epsilon > 0$ there exists a periodic point $p \in \mathcal{M}$ for which the Lyapunov exponents $\lambda_1(p) \leq \cdots \leq \lambda_d(p)$ of $\mathcal{A}$ satisfy $|\lambda_i - \lambda_i^{(p)}| < \epsilon$ for $i = 1, \ldots, d$. 

3.2. The largest and smallest Lyapunov exponents. We are primarily interested in
the largest and the smallest Lyapunov exponents of \( A \) with respect to \( \mu \), which can be
expressed as follows:

\[
\lambda_+(A, \mu) = \lambda_1 = \lim_{n \to \infty} \frac{1}{n} \log \| A^n_x \| \quad \text{for } \mu\text{-a.e. } x \in \mathcal{M}, \\
\lambda_-(A, \mu) = \lambda_1 = \lim_{n \to \infty} \frac{1}{n} \log \| (A^n_x)^{-1} \|^{-1} \quad \text{for } \mu\text{-a.e. } x \in \mathcal{M}.
\]

These limits exist and are constant almost everywhere since the sequence of functions
\( a_n(x) = \frac{1}{n} \log \| A^n_x \| \) is subadditive, i.e. it satisfies

\[
a_{n+k}(x) \leq a_n(x) + a_k(f^k x) \quad \text{for all } x \in \mathcal{M} \text{ and } n, k \in \mathbb{N}.
\]

Hence by the Subadditive Ergodic Theorem for \( \mu\)-a.e. \( x \),

\[
\lim_{n \to \infty} \frac{a_n(x)}{n} = \lim_{n \to \infty} \frac{a_n(\mu)}{n} = \inf \left\{ \frac{a_n(\mu)}{n} : n \in \mathbb{N} \right\}, \quad \text{where } a_n(\mu) = \int_X a_n(x) \, d\mu.
\]

Now we briefly consider the infinite-dimensional case, where each fiber \( E_x \) is a Banach space \( V \). In this case, there is no Oseledets MET in general, but upper and lower Lyapunov exponents \( \lambda_+(A, \mu) \) and \( \lambda_-(A, \mu) \) can still be defined as in (3.1).

In infinite-dimensional setting, it is not always possible to approximate \( \lambda_+(A, \mu) \) and \( \lambda_-(A, \mu) \) by Lyapunov exponents at periodic points. The following proposition is based on an example by L. Gurvits of a pair of operators whose joint spectral radius is greater than the generalized spectral radius [Gu95].

**Proposition 3.3.** There exists a locally constant cocycle \( A \) over a full shift on two
symbols and an ergodic invariant measure \( \mu \) such that \( \lambda(A, \mu) > \sup_{\mu_p} \lambda(A, \mu_p) \),
where the supremum is taken over all uniform measures \( \mu_p \) on periodic orbits.

In the setup of Theorem 3.2, the largest and smallest exponents of \( A \) with respect to \( \mu \) can also be approximated by \( \frac{1}{n} \log \| A^n_x \| \) and \( \frac{1}{n} \log \| (A^n_x)^{-1} \|^{-1} \), respectively [K11]. We note that for periodic measures \( \mu_p \),

\[
\lambda_+(A, \mu_p) = (1/n) \log (\text{spectral radius of } A^n_p) \leq (1/n) \log \| A^n_p \|,
\]

and the inequality can be strict. The approximation of \( \lambda_+(A, \mu) \) and \( \lambda_-(A, \mu) \) in terms
of the norms extends to the infinite-dimensional case.

**Theorem 3.4.** [KS17] Let \( (\mathcal{M}, f) \) be a hyperbolic system, let \( \mu \) be an ergodic \( f \)-invariant
Borel probability measure on \( \mathcal{M} \), and let \( A \) be a Hölder continuous \( GL(V) \)-valued cocycle
over \( f \), where \( V \) is a Banach space.

Then for any \( \epsilon > 0 \) there exists a periodic point \( p = f^np \) such that

\[
\left| \lambda_+(A, \mu) - \frac{1}{n} \log \| A^n_p \| \right| < \epsilon \quad \text{and} \quad \left| \lambda_-(A, \mu) - \frac{1}{n} \log \| (A^n_p)^{-1} \|^{-1} \right| < \epsilon.
\]
3.3. Cocycles with one exponent and Continuous Amenable Reduction.

Now we return to the finite-dimensional case and consider cocycles with one Lyapunov exponent, i.e. satisfying \( \lambda_+ (\mathcal{A}, \mu) = \lambda_- (\mathcal{A}, \mu) \). Clearly, this is a broader class than conformal or uniformly quasiconformal cocycles.

When \( \mathcal{A} \) has more than one Lyapunov exponent, the invariant sub-bundles \( E^i \) given the Oseledets MET are measurable but not necessarily continuous. The next theorem establishes continuity of measurable invariant sub-bundles for fiber bunched cocycles with one Lyapunov exponent. The following theorem is a corollary of results by A. Avila and M. Viana in [AV10].

**Theorem 3.5.** Let \( f \) be a transitive Anosov diffeomorphism, \( \mathcal{A} : \mathcal{E} \to \mathcal{E} \) be a \( \beta \)-Hölder fiber bunched linear cocycle over \( f \), and \( \mu \) be an ergodic \( f \)-invariant probability measure with full support and local product structure. If \( \lambda_+ (\mathcal{A}, \mu) = \lambda_- (\mathcal{A}, \mu) \) then any \( \mu \)-measurable \( \mathcal{A} \)-invariant sub-bundle of \( \mathcal{E} \) is \( \beta \)-Hölder.

Now we obtain a structure theorem for cocycles with one exponent.

**Theorem 3.6.** [KS13] Let \( f \) be a transitive Anosov diffeomorphism and \( \mathcal{A} : \mathcal{E} \to \mathcal{E} \) be a \( \beta \)-Hölder linear cocycle over \( f \). Suppose that for every periodic point \( p = f^n p \) the invariant measure \( \mu_p \) on its orbit satisfies \( \lambda_+ (\mathcal{A}, \mu_p) = \lambda_- (\mathcal{A}, \mu_p) \), i.e. all eigenvalues of \( \mathcal{A}_p^n \) are equal in modulus.

Then there exist a flag of \( \beta \)-Hölder \( \mathcal{A} \)-invariant sub-bundles

\[
\{0\} = E^0 \subset E^1 \subset \ldots \subset E^{j-1} \subset E^j = \mathcal{E}
\]

and \( \beta \)-Hölder Riemannian metrics on the factor bundles \( E^i / E^{i-1}, \ i = 1, \ldots, j \), so that the factor-cocycles induced by \( \mathcal{A} \) on \( E^i / E^{i-1} \) are conformal. Moreover, there exists a positive \( \beta \)-Hölder function \( \phi : \mathcal{M} \to \mathbb{R} \) s.t. the factor-cocycles of \( \phi \mathcal{A} \) on \( E^i / E^{i-1} \) are isometries.

In the case when \( E_1 = E \), the cocycle \( \mathcal{A} \) itself is conformal on \( E \) with respect to some continuous Riemannian metric.

**Outline of the proof.** We note that the periodic assumptions imply fiber bunching and one exponent for each ergodic measure. We take \( \mu \) to be the measure of maximal entropy, or the invariant volume if it exists, and trivialize the bundle \( \mathcal{E} \) on a set of full measure, i.e. we measurably identify \( \mathcal{E} \) with \( \mathcal{M} \times \mathbb{R}^d \) and view \( \mathcal{A} \) as a function \( \mathcal{M} \to GL(d, \mathbb{R}) \). Thus we can use Zimmer’s Amenable Reduction:

**Theorem 3.7.** (Zimmer’s Amenable Reduction) Let \( f \) be an ergodic transformation of a measure space \( (X, \mu) \) and let \( \mathcal{A} : X \to GL(d, \mathbb{R}) \) be a measurable function.

Then there exists a measurable function \( C : X \to GL(d, \mathbb{R}) \) such that the function \( B(x) = C^{-1} (fx) A(x) C(x) \) takes values in an amenable subgroup of \( GL(d, \mathbb{R}) \).

Thus we obtain a measurable coordinate change function \( C : \mathcal{M} \to GL(d, \mathbb{R}) \) such that \( B(x) = C^{-1} (fx) A_x C(x) \) is \( \mu \)-a.e. in an amenable subgroup \( G \subset GL(d, \mathbb{R}) \).
Amenable subgroups of Lie groups were studied in [M79]. There are $2^{d-1}$ standard maximal amenable subgroups of $GL(d, \mathbb{R})$. They correspond to the distinct compositions of $d$, $d_1 + \cdots + d_k = d$, and each group consists of all block-triangular matrices of the form

\[
\begin{bmatrix}
B_1 & * & \cdots & * \\
0 & B_2 & \ddots & \\
\vdots & \ddots & \ddots & * \\
0 & \cdots & 0 & B_k
\end{bmatrix}
\]

where each diagonal block $B_i$ is a $d_i \times d_i$ conformal matrix, i.e. a scalar multiple of an orthogonal matrix. Any amenable subgroup of $GL(d, \mathbb{R})$ has a finite index subgroup which is contained in a conjugate of one of these standard subgroups. Thus we may assume that $G$ has a finite index subgroup $G_0$ which is contained in one of the standard subgroups.

We concentrate on the simplest case when $G_0 = G$. Then the sub-bundle $V^i$ spanned by the first $d_1 + \cdots + d_i$ coordinate vectors in $\mathbb{R}^d$ is $B$-invariant for $i = 1, \ldots, k$. Denoting $E^i_x = C(x)V^i$ we obtain the corresponding flag of measurable $A$-invariant sub-bundles

$E^1 \subset E^2 \subset \cdots \subset E^k = E$ with $\dim E^i = d_1 + \cdots + d_i$.

By Theorem 3.5 we may assume that the sub-bundles $E^i$ are Hölder continuous. Since $B_1(x)$ is a conformal matrix for $\mu$-a.e. $x$, the push forward by $C$ of the standard conformal structure on $V^1$ is invariant under the restriction of $A$ to $E^1$ and hence Hölder continuous.

Similarly, we consider the factor-bundles $E^i/E^{i-1}$ over $\mathcal{M}$ with the natural induced cocycle $A^{(i)}$. Since the matrix of the map induced by $B$ on $V^i/V^{i-1} = \mathbb{R}^{d_i}$ is $B_i$, it preserves the standard conformal structure on $\mathbb{R}^{d_i}$. Pushing it forward by $C$ we obtain a measurable conformal structure $\tau_i$ on $E^i/E^{i-1}$ invariant under $A^{(i)}$. The holonomies $H^s$ and $H^u$ induce holonomies for $A^{(i)}$ on $E^i/E^{i-1}$. As before, we conclude that $\tau_i$ is essentially invariant under these holonomies and hence is Hölder continuous on $\mathcal{M}$.

Now we outline the argument for the case when $G_0 \neq G$. An example illustrating this case is when $G_0$ is the full diagonal subgroup and $G$ is its normalizer, which is the finite extension of $G_0$ that contains all permutations of the coordinate axes.

In the general case, $G_0$ still has the invariant flag $V^1 \subset V^2 \subset \cdots \subset V^k = \mathbb{R}^d$ with conformal structures on the factors. The elements of $G$ may not preserve this flag, instead there are several images of it under $G$:

$V^1_{(1)} \subset V^2_{(1)} \subset \cdots \subset V^k_{(1)}$

$\ldots$

$V^1_{(\ell)} \subset V^2_{(\ell)} \subset \cdots \subset V^k_{(\ell)}$

These flags are again mapped by $C$ to the corresponding measurable flags on $E$ as before. The sub-bundles in the flags are not $A$-invariant individually, but $A$ preserves the union of subspaces on each level of the flag $C(V^i_{(1)}) \cup \cdots \cup C(V^i_{(\ell)})$, $i = 1, \ldots, k-1$. For each level of the flag we show that the union is essentially holonomy invariant and hence is Hölder continuous. The union may not split into individual invariant sub-bundles since it
may “twist” along non-trivial loops in $\mathcal{M}$. Thus to obtain actual continuous sub-bundles one needs in general to pass to a special finite cover of $\mathcal{M}$. These sub-bundles are not necessarily invariant as $A$ may permute them, but they are invariant under an iterate of $A$. In this way we can obtain the general result below, Theorem 3.9

To complete the proof of Theorem 3.6, we claim that the assumption on the periodic data implies that $A$ is conformal on the sub-bundle spanned by the union $C(V_1^1) \cup \cdots \cup C(V_1^\ell)$, and then similarly for the induced cocycles on the factors. Indeed, the scalar cocycles that give expansion/contraction of $A$ (more precisely, of an iterate of the lift of $A$ as above) are continuously cohomologous by the Livsic Periodic Point Theorem, since the exponents at each periodic point are the same. This yields uniform quasiconformality and hence conformality of $A$ on the span. The same reasoning shows that the scalar cocycles of expansion/contraction for the different conformal factors of $A$ are all cohomologous to one scalar function, whose inverse gives the function $\phi$ in the theorem. □

Corollary 3.8. (Polynomial growth of quasiconformal distortion and norm)

Let $f$ be a transitive Anosov diffeomorphism. Suppose that for every $f$-periodic point $p$ the invariant measure $\mu_p$ on its orbit satisfies $\lambda_+(A, \mu_p) = \lambda_-(A, \mu_p)$. Then there exists $m < \dim \mathcal{E}_x$ and $C$ such that

$$Q_A(x, n) \leq Cn^{2m} \text{ for all } x \in \mathcal{M} \text{ and } n \in \mathbb{Z}. $$

Moreover, if $\lambda_+(A, \mu_p) = \lambda_-(A, \mu_p) = 0$ for every $\mu_p$, then there exists $m < \dim \mathcal{E}_x$ and $C$ such that

$$\|A^n_x\| \leq C|n|^m \text{ for all } x \in \mathcal{M} \text{ and } n \in \mathbb{Z}. $$

One can take $m = j - 1$, which is the number of non-trivial sub-bundles in (3.2).

3.4. The theorem for partially hyperbolic diffeomorphisms. As we indicated above, the argument for the proof of Theorem 3.6 yields a general result, which holds also for partially hyperbolic $f$. In the next theorem we assume one exponent for the volume and fiber bunching of $A$. The assumptions of Theorem 3.6 imply one exponent for each ergodic measure as well as fiber bunching. For hyperbolic $f$, the invariant volume $\mu$ can be replaced by any ergodic $f$-invariant probability measure with full support and local product structure, and the resulting conformal structures and sub-bundles are Hölder continuous.

Theorem 3.9. [KS13] Let $f$ be an accessible partially hyperbolic center bunched diffeomorphism preserving a volume $\mu$ and let $A : \mathcal{E} \to \mathcal{E}$ be a Hölder continuous linear cocycle over $f$. Suppose that $A$ is fiber bunched and $\lambda_+(A, \mu) = \lambda_-(A, \mu)$.

Then there exists a finite cover $\tilde{A} : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ of $A$ and $N \in \mathbb{N}$ such that $\tilde{A}^N$ satisfies the following property. There exist a flag of continuous $\tilde{A}^N$-invariant sub-bundles

\begin{equation}
\{0\} = \tilde{\mathcal{E}}^0 \subset \tilde{\mathcal{E}}^1 \subset \cdots \subset \tilde{\mathcal{E}}^{k-1} \subset \tilde{\mathcal{E}}^k = \tilde{\mathcal{E}}
\end{equation}

and continuous conformal structures on the factor bundles $\tilde{\mathcal{E}}^i/\tilde{\mathcal{E}}^{i-1}$, $i = 1, \ldots, k$, invariant under the factor-cocycles induced by $\tilde{A}^N$. 

The proof shows that when it is necessary to pass to a cover, the resulting cocycle $\tilde{A}^N$ preserves more than one flag as in (3.4). Their union is preserved by $\tilde{A}$ and is the lift of an invariant object for $A$. To illustrate this we construct a cocycle $A$ on $E = T^2 \times \mathbb{R}^2$ with no invariant $\mu$-measurable sub-bundles or conformal structures. Its lift $\tilde{A}$ to a double cover preserves two continuous line bundles, while $A$ preserves a continuous field of pairs of lines.

References


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