Math 112 (Calculus I)
Final Exam Form A KEY

Multiple Choice. Fill in the answer to each problem on your scantron. Make sure your name, section and instructor is on your scantron.

1. Simplify $\sin(\tan^{-1} \dfrac{5}{12})$:

   a) 1 b) $\dfrac{4}{5}$ c) $\dfrac{3}{4}$

   d) $\dfrac{5}{13}$ e) $\dfrac{12}{13}$ f) 3

   g) None of the above.

   Solution: d)

2. Find $\lim_{x\to 1} \dfrac{x^2+x+2}{2x^2+x-3}$

   a) Does not exist b) -1 c) 2

   d) $\dfrac{1}{2}$ e) 4 f) $\dfrac{1}{4}$

   g) None of the above.

   Solution: a)

3. If $f(x) = Ax^2 + Bx + C$, find $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$.

   a) $Ax^2 + Bx$ b) $2Ax + B$ c) $2Ax^2 + Bx + C$

   d) $2Ax^2 + Bx$ e) 0 f) Does not exist.

   Solution: b)

4. The derivative of $h(x) = x^{2x}$ is

   a) $2x^{2x-1}$ b) $2x \cdot x^{2x-1}$ c) $x^{2x} \ln x$

   d) $2x^{2x}(1 + \ln x)$ e) $2x^{2x}$

   Solution: d)

5. The equation of the tangent line to

$$y = \sin^2 x$$

at $(\frac{\pi}{6}, \frac{1}{4})$ is

a) $y = \frac{1}{4} + 2(x - \frac{\pi}{6}) \sin x \cos x$ b) $y = \frac{\sqrt{3}}{2}x + \frac{3 - \pi\sqrt{3}}{12}$ c) $y = \frac{1}{4}$

d) $y = \frac{1}{2}x + \frac{3 - \pi}{12}$ e) None of the above.

6. For the function $f(x) = x^2$, for which value of c is $f'(c)$ equal to the average rate of change of f on $[0, 3]$?

a) 0 b) $\frac{3}{2}$ c) 3

d) $\frac{5}{2}$ e) 6 f) There is no such c.

g) None of the above.

Solution: b)

7. A farmer wishes to fence in 100 square feet of area into two adjoining rectangular regions. The outside fence costs \$8 per linear foot and the dividing fence inside costs \$9 per linear foot. What is the minimal cost to fence the region?

a) \$ 100 b) \$ 200 c) \$ 300

d) \$ 400 e) \$ 600 f) None of the above.

Solution: d)

8. $\int_0^1 x\sqrt{1 + 3x^2}\, dx =$

a) $\frac{1}{9}$ b) $\frac{7}{9}$ c) $\frac{7}{3}$

d) 7 e) 10 f) None of the above.

Solution: b)

Short Answer. Fill in the blank with the appropriate answer.

9. (10 points)

(a) Let $f(x) = c \cdot a^x$ be an exponential function, where $f(1) = 3$ and $f(2) = 12$. Write out the equation for $f(x)$, replacing a and c by their actual values. $f(x) = \underline{\frac{3}{4}4^x}$

(b) Simplify completely to a single expression: $2\ln(b) - 3\ln(a) + 4\ln(2) + \ln(3) = \underline{\ln\left(\frac{48b^2}{a^3}\right)}$

(c) $\frac{d}{dx}(\ln(\cosh x)) = \underline{\frac{\sinh x}{\cosh x} = \tanh x}$

(d) $\frac{d}{dx}(\pi^2 + e^3) = \underline{0}$

(e) $\frac{d}{dx}\ln(2x+1) = \underline{\frac{2}{2x+1}}$

(f) The equation of the tangent line to the curve $y = xe^{x^2}$ at (0,0) is $\underline{y = x}$.

(g) Use linear approximation to estimate $\sqrt[3]{8.006}$:$\underline{2.0005}$

(h) If $f'(x) = 3x^2 + 2x - \frac{5}{x}$, then $\underline{f(x) = x^3 + 2x - 5\ln|x| + C}$

(i) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^2} \cdot \frac{1}{n}$ represents what definite integral? $\underline{\int_0^1 x^2\, dx}$

(j) $\int_1^9 \frac{1}{x^{3/2}}\, dx = \underline{\frac{4}{3}}$

Free Response. For problems 10 - 17, write your answers in the space provided. Use the back of the page if needed, indicating that fact. Neatly show all work.

10. (8 points)

(a) State the definition of the derivative.

Solution: Either

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

or

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is acceptable.

(b) Find $f'(2)$ where $f(x) = \dfrac{1}{x-1}$ **using the definition of the derivative**.

Solution:

Version A:

$$f'(2) = \lim_{x \to 2} \frac{\frac{1}{x-1} - 1}{x - 2}$$

$$= \lim_{x \to 2} \frac{1 - (x-1)}{(x-2)(x-1)} = \lim_{x \to 2} \frac{-1}{x-1} = -1.$$

Version B:

$$f'(-1) = \lim_{x \to -1} \frac{\frac{1}{x-1} - \left(\frac{-1}{2}\right)}{x - (-1)}$$

$$= \lim_{x \to -1} \frac{2 + (x-1)}{2(x-1)(x+1)} = \lim_{x \to -1} \frac{1}{2(x-1)} = -\frac{1}{4}.$$

11. (8 points) Given the equation

$$x^6 + 3xy + y^5 = 5,$$

find y' at the point $(1, 1)$.

Solution: We differentiate implicitly:

Version A:

$$6x^5 + 3y + 3xy' + 5y^4 y' = 0$$

Substituting 1 for x and 1 for y, we have

$$9 + 8y' = 0,$$

or $y' = -\dfrac{9}{8}$.

Version B:

$$6x^5 + 2y + 2xy' + 10y^4 y' = 0$$

$$8 + 12y' = 0$$

$$y' = -\frac{2}{3}.$$

12. (8 points) The volume of a sphere is increasing at a rate of 3 cubic centimeters per second. How fast is the radius increasing when the radius is 10 centimeters?

Solution:

$$V = \frac{4}{3}\pi r^3$$
$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Version A:

$$3 = 4\pi(100)\frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{3}{400\pi} \text{ cm/s}$$

Version B:

$$2 = 4\pi(81)\frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{1}{162\pi} \text{ cm/s}$$

13. (8 points) Find all local extrema of the function

$$f(x) = (x^2 - 3)e^x,$$

and classify them as local maxima or local minima.

Solution:

$$f'(x) = 2xe^x + (x^2 - 3)e^x = (x^2 + 2x - 3)e^x = (x + 3)(x - 1)e^x.$$

Thus, $x = -3$ and $x = 1$ are critical points.

$$f''(x) = (2x + 2)e^x + (x^2 + 2x - 3)e^x = (x^2 + 4x - 1)e^x.$$
$$f''(-3) = (9 - 12 - 1)e^{-3} < 0$$
$$f''(1) = (1 + 4 - 1)e^1 > 0.$$

Thus, f has a maximum of $f(-3) = 6e^{-3}$ and a minimum of $f(1) = -2e$.

For questions 14 to 15, find the limits.

14. (8 points) $\lim_{x\to\infty} \frac{\ln(\ln x)}{3x}$

Solution: We use L'Hopital's rule:

$$\lim_{x\to\infty} \frac{\ln(\ln x)}{3x} = \lim_{x\to\infty} \frac{\frac{1}{x\ln x}}{3} = \lim_{x\to\infty} \frac{1}{3x\ln x} = 0.$$

15. (8 points) $\lim_{x\to\infty} x^2 \ln(1 - \frac{1}{x})$

Solution: Again, using L'hopital's rule,

$$\lim_{x\to\infty} x^2 \ln(1 - \frac{1}{x}) = \lim_{x\to\infty} \frac{\ln(1-\frac{1}{x})}{\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1-\frac{1}{x}} \cdot \frac{1}{x^2}}{-\frac{2}{x^3}} = \lim_{x\to\infty} \frac{x}{-2(1-\frac{1}{x})} = \infty.$$

16. (9 points) Find the following:

(a) $\int \left(x^3 + x^{-3}\right)\, dx$

Solution:

Version A:

$$\frac{x^4}{4} - \frac{1}{2x^2} + C$$

Version B:

$$\frac{x^5}{5} - \frac{1}{x} + C$$

(b) $\int \frac{1+\cos^2 x}{\cos^2 x}\, dx$

Solution:

Version A:

$$\int \sec^2 x + 1\, dx = \tan x + x + C$$

Version B:

$$\int \frac{1}{t^2} - \frac{1}{t}\, dt = -\frac{1}{t} - \ln|t| + C$$

(c) $\int_0^2 x(\sqrt[3]{x} + \sqrt[5]{x})\, dx$

Solution:

Version A:

$$\int_0^2 x^{4/3} + x^{6/5}\, dx = (\frac{3}{7}x^{7/3} + \frac{5}{11}x^{11/5})_0^2 = \frac{12}{7}2^{1/3} + \frac{20}{11}2^{1/5}.$$

Version B:

$$\int_0^2 x^{5/3} + x^{6/5}\, dx = (\frac{3}{8}x^{8/3} + \frac{5}{11}x^{11/5})_0^2 = \frac{3}{2}2^{2/3} + \frac{20}{11}2^{1/5}.$$

17. (9 points) Sketch the curve of

$$f(x) = \frac{x-1}{x^2},$$

labelling all asymptotes, intercepts, local max and mins, and inflection points.

Solution: **Note:** This problem can also be done by rewriting the function as

$$f(x) = x^{-1} - x^{-2}.$$

the solution will be given without the simplification in case some students do it that way.

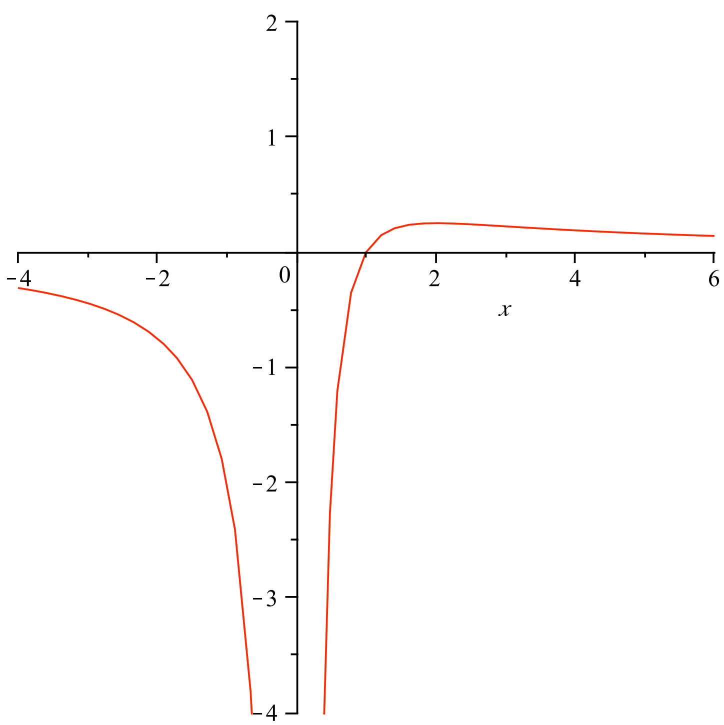
Clearly, the function has a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 0$. Also, $(1, 0)$ is the only intercept.

$$f'(x) = \frac{x^2 - 2x(x-1)}{x^4} = \frac{2-x}{x^3}.$$

The only critical point, therefore is $x = 2$.

$$f''(x) = \frac{-x^3 - (2-x)3x^2}{x^6} = \frac{2x-6}{x^4}$$

Notice that $f''(2) < 0$, so we have a local maximum at $(2, \frac{1}{4})$. Also, $f''(3) = 0$. Since $f''(-1) < 0$, f is concave down on $(-\infty, 0)$. As $f''(2) < 0$, f is concave down on $(0, 3)$. As $f''(4) > 0$, f is concave up on $(3, \infty)$. Thus, $(3, \frac{2}{9})$ is an inflection point.



END OF EXAM