Math 112 (Calculus I)
Final Exam Form A KEY

21. (a) (4 points) Find $\lim_{x\to\infty} \sqrt{x^2+1} - x$.

Solution:

$$\begin{aligned}\lim_{x\to\infty} \sqrt{x^2+1} - x &= \lim_{x\to\infty} \frac{(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)}{\sqrt{x^2+1}+x} \\ &= \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x} = 0.\end{aligned}$$

(b) (4 points) Find $\lim_{x\to\infty} \frac{x\ln(x)}{x^2}$.

Solution:

$$\lim_{x\to\infty} \frac{x\ln(x)}{x^2} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

using L'Hopital's rule.

22. (5 points) Compute the definite integral $\int_1^2 \frac{x}{\sqrt{9+x^2}}\,dx$.

Solution:

Use $u = 9 + x^2$. Then, $du = 2x\,dx$, and

$$\int_1^2 \frac{x}{\sqrt{9+x^2}}\,dx \int_{10}^{13} \frac{1}{2}u^{-1/2}\,du = u^{1/2}|_{10}^{13} = \sqrt{13} - \sqrt{10}$$

23. (5 points) You own an emu ranch, and want to build a rectangular fence around your herd. One side of the fence must be built along the side of a river so the emu's have access to fresh water. You want to enclose at least 1000 square meters. The fencing material costs 6 dollars per meter, but the fencing along the river costs 10 dollars per meter (to allow the emu to drink the water). What are the dimensions which minimize the cost of your fence? What is the cost?

Solution: Let x be the length along the river while y is the length perpendicular to it. The total cost is

$$C = 16x + 12y.$$

Since the area is 1000, we have

$$xy = 1000, \text{ or } y = \frac{1000}{x}.$$

Hence,

$$C = 16x + \frac{12000}{x}.$$

Notice

$$C' = 16 - \frac{12000}{x^2} = \frac{16x^2 - 12000}{x^2}.$$

A critical point is at

$$16x^2 - 12000 = 0 \text{ or } x^2 = 750.$$

Thus, $x = 5\sqrt{30}$ feet, and

$$y = \frac{1000}{5\sqrt{30}} = \frac{200}{\sqrt{30}}.$$

Notice this is a minimum because

$$C' = \frac{24000}{x^3},$$

which is positive for $x > 0$, and there is only one critical point in $(0, \infty)$.

24. (5 points) Find the derivative of $f(x) = (2x)^{\ln(x)}$.

Solution:

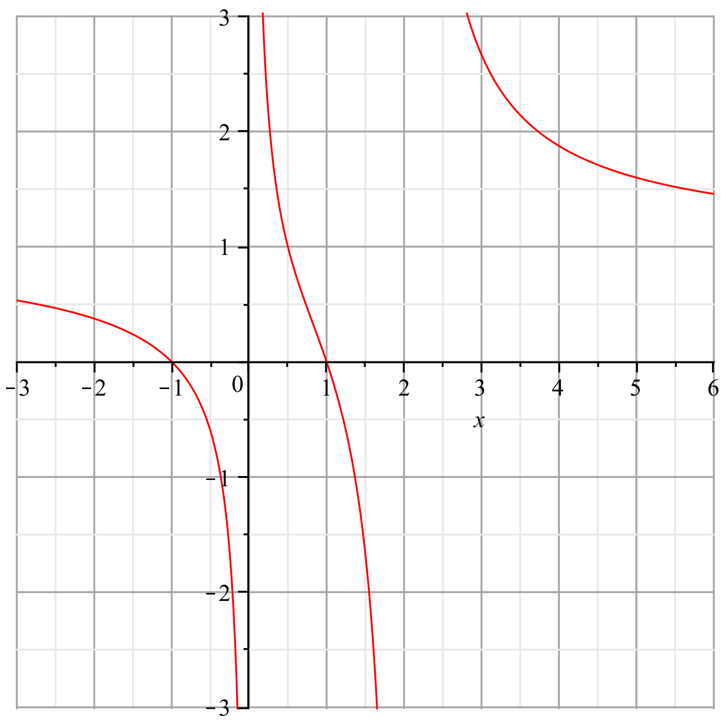
We use logarithmic differentiation.

$$\ln(f(x)) = \ln(x)\ln(2x).$$

$$\frac{1}{f(x)}f'(x) = \frac{1}{x}\ln(2x) + \frac{1x}{\ln}x.$$

$$f'(x) = \frac{(2x)^{\ln(x)}}{x}(\ln(2x) + \ln x).$$

25. (6 points) Sketch the curve $f(x) = \dfrac{(x^2-1)}{x(x-2)}$. Label all asymptotes, intercepts, local maxs and mins, and any inflection points.



Solution:

The asymptotes are at $x = 0$, $x = 2$, and $y = 1$. The x intercepts are at -1 and 1, and there are no y intercepts. There are no critical points, and hence no max and mins.

26. (a) (3 points) Using the definition of derivative, write a limit (without evaluating the limit) that gives the derivative of $f(x) = 1/(x-1)$.

Solution:

$$f'(x) = \lim_{h\to 0}\frac{\frac{1}{x+h-1} - \frac{1}{x-1}}{h}.$$

(b) (3 points) Evaluate the limit you gave in part (a).

Solution:

$$f'(x) = \lim_{h\to 0}\frac{\frac{1}{x+h-1} - \frac{1}{x-1}}{h} = \lim_{h\to 0}\frac{x-1-(x+h-1)}{(x+h-1)(x-1)h}$$

$$= lim_{h\to 0}\frac{-h}{(x+h-1)(x-1)h} = \lim_{h\to 0}\frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}.$$

27. (5 points) The area of a circle is increasing at a rate of 3 m/s^2. When the area is $\pi\,\mathrm{m}^2$, how fast is the circumference growing?

Solution:

$$A = \pi r^2,$$

so

$$\frac{dA}{dt} = 2\pi r\,\frac{dr}{dt}.$$

Since the rate of change of the area is 3,

$$\frac{dr}{dt} = \frac{3}{2\pi}.$$

Since the circumference is

$$C = 2\pi r,$$

$$\frac{dC}{dt} = 2\pi\frac{dr}{dt},$$

so

$$\frac{dC}{dt} = 3.$$

28. (5 points) The equation $e^{x+y} = y^2$ implicitly defines y as a function of x. Find dy/dx.

Solution:

$$(1+y')e^{x+y} = 2yy'$$

$$(e^{x+y} - 2y)y' = -e^{x+y}$$

$$y' = \frac{e^{x+y}}{2y - e^{x+y}}.$$

29. (5 points) Find the equation of the tangent line to the curve $y = e^x(x^2 + \ln(x+1) + 3)$ at the point $(0, 3)$.

Solution:

$$y' = e^x(x^2 + \ln(x+1) + 3) + e^x(2x + \frac{1}{x+1})$$

$$y'(0) = 1 \cdot 3 + 1 \cdot 1 = 4.$$

$$y - 3 = 4x$$

$$y = 4x + 3$$

END OF EXAM