Math 112 (Calculus I)
Final Exam Form A KEY

Part I: Multiple Choice. Enter your answer on the scantron. Work will not be collected or reviewed.

Free response: Write your answer in the space provided. Answers not placed in this space will be ignored.

21. (8 points) **Short answer.** Two points each part. You do not need to show your work on this problem.

(a) Given $\epsilon > 0$, and the statement

$$\lim_{x \to -1} (-2x + 3) = 5,$$

find the largest $\delta > 0$ so that the following statement is true:

$$\text{If } 0 < |x + 1| < \delta, \text{ then } |-2x + 3 - 5| < \epsilon.$$

Solution: $\epsilon/2$

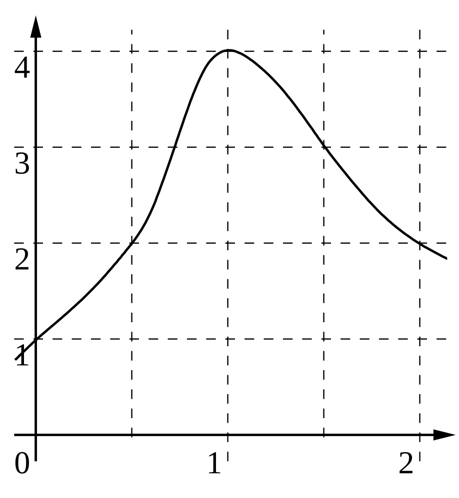
(b) Find $\displaystyle\lim_{x \to \infty} \frac{5 - 3x^3}{\sqrt{81x^6 - 16}}$.

Solution: $-1/3$

(c) Evaluate the integral $\displaystyle\int (1 + x^2)\, dx$.

Solution: $x + x^3/3 + C$

(d) Let $f(x)$ be the function whose graph is shown below. Use *right* hand sums with four rectangles to estimate $\displaystyle\int_0^2 f(x)dx$.



Solution: $11/2 = 5.5$

22. (5 points) Use the definition of the derivative to show:

$$\text{If } f(x) = \frac{1}{x - 1}, \text{ then } f'(x) = -\frac{1}{(x - 1)^2}.$$

No credit will be given if a method besides the definition of the derivative is used.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{1}{x+h-1}-\frac{1}{x-1}}{h} \\ &= \lim_{h\to 0}\frac{x-1-x-h+1}{h(x+h-1)(x-1)} = \lim_{h\to 0}\frac{-h}{h(x+h-1)(x-1)} \\ &= \lim_{h\to 0}\frac{-1}{(x+h-1)(x-1)} = -\frac{1}{(x-1)^2}. \end{aligned}$$

23. (4 points) Find the linear approximation of the function $f(x) = x^{3/2}$ at $a = 100$, and use it to approximate the number $\sqrt{(101)^3}$.

Solution: The formula for linear approximation is given by $L(x) = f(a) + f'(a)(x - a)$.

First, $f(a) = f(100) = (100)^{3/2} = 10^3 = 1000$.

Next, $f'(x) = \frac{3}{2}x^{1/2}$, so $f'(a) = f'(100) = \frac{3}{2}(100)^{1/2} = 15$.

Then the linear approximation of $f(x)$ is given by

$$L(x) = 1000 + 15(x - 100).$$

So $\sqrt{(101)^3} \approx 1000 + 15(101 - 100) = 1015$.

24. (7 points) Prove that $f(x) = 2x + \sin x$ has at most one root.

Solution: Suppose that $f(x)$ has two roots, say there are numbers a and b such that $f(a) = 0 = f(b)$. Since f is continuous and differentiable everywhere, it is continuous on $[a, b]$ and differentiable on (a, b). Then we have the hypotheses of Rolles' Theorem, so by Rolles' Theorem there is a number c such that $f'(c) = 0$.

On the other hand, $f'(x) = 2 + \cos x$. Since $-1 \le \cos x \le 1$, $2 + \cos x$ can never equal 0. This is a contradiction.

Hence $f(x)$ has at most one root.

25. (7 points) Find the limit. $\lim_{x\to\infty}\left(\frac{1}{x^2}\right)^{1/x}$

Solution: If we let $L = \lim_{x\to\infty}\left(\frac{1}{x^2}\right)^{1/x}$, then

$$\ln L = \lim_{x\to\infty}\ln\left(\frac{1}{x^2}\right)^{1/x} = \lim_{x\to\infty}\frac{1}{x}\ln\left(\frac{1}{x^2}\right) = \lim_{x\to\infty}\frac{\ln(1)-\ln(x^2)}{x} = \lim_{x\to\infty}\frac{-2\ln(x)}{x}$$

Note that the limit on the far right is indeterminate of type $\frac{\infty}{\infty}$, so we apply L'Hospital's rule to obtain:

$$\ln L = \lim_{x\to\infty}\frac{\frac{-2}{x}}{1} = 0$$

So $L = e^0 = 1$.

26. (8 points) Suppose the area of a right triangle is 18 cm^2. Find the smallest possible length of its hypoteneuse.

Solution: For a right triangle, let b denote the base and a the height. Then the area is given by

$$A = \frac{1}{2}ab,$$

and the square of the hypoteneuse is given by:

$$h^2 = a^2 + b^2.$$

We know that $A = 18$ (cm^2), hence $a = 36/b$. Substitute this into our formula for h^2:

$$h^2 = \frac{(36)^2}{b^2} + b^2.$$

Now, notice that h^2 and h will both have a minimum at the same values of a and b, so we can save ourselves a little algebra and minimize $H(b) = \frac{(36)^2}{b^2} + b^2$ rather than its square root (although using the square root will give the same answer).

We find the derivative:

$$H'(b) = -2\frac{(36)^2}{b^3} + 2b$$

Set it equal to zero and solve for b:

$$-2\frac{(36)^2}{b^3} + 2b = 0 \quad \Leftrightarrow \quad -2(36)^2 + 2b^4 = 0 \quad \Leftrightarrow \quad b^4 = (36)^2 \quad \Leftrightarrow \quad b = 6 \text{ (cm)}$$

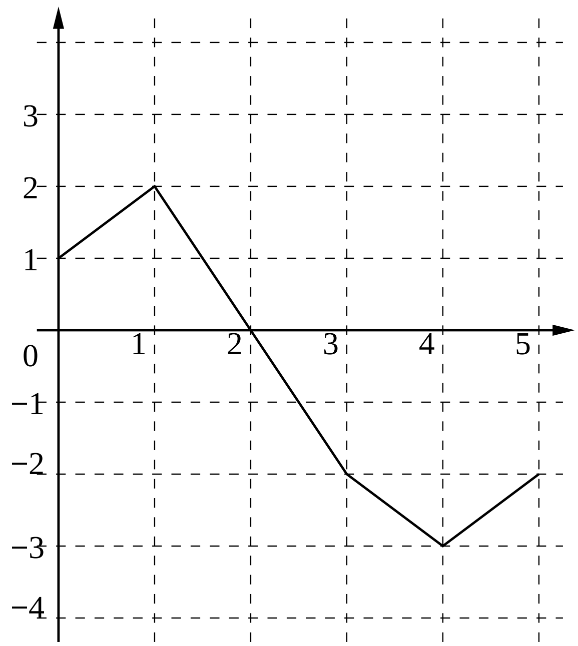
Use the second derivative test to check that this really gives a minimum:

$$H''(b) = 6\frac{(36)^2}{b^4} + 2 > 0 \text{ when } b = 6.$$

So $b = 6$ is a minimum, and $H(6) = \frac{(36)^2}{36} + 36 = 72$. Hence the shortest possible hypoteneuse is the square root of this, or

$$6\sqrt{2} \text{ cm}.$$

27. (8 points) The graph of a function $f(x)$ is shown. Let $g(x) = \int_0^x f(t)\,dt$, for $0 \le x \le 5$.



(a) Evaluate $g(0)$, $g(2)$, and $g(5)$.
Solution: $g(0) = 0, g(2) = 2.5, g(5) = -3.5$

(b) Where is $g(x)$ increasing on $[0, 5]$? Decreasing?
Solution: Increasing on $(0, 2)$, decreasing on $(2, 5)$.

(c) Sketch the graph of $g(x)$ on the same axes of $f(x)$, labeling all local maxima and minima.
Solution: Graph should include the three points $(0, 0)$, $(2, 2.5)$, and $(5, -3.5)$. It should be increasing from $(0, 0)$ to $(2, 2.5)$, then decreasing from $(2, 2.5)$ to $(5, -3.5)$. The point $(2, 2.5)$ is a local maximum. There are no local minima.

28. (7 points) A pendulum swings with velocity $v(t) = \cos t$. Find the total distance the pendulum travels over $0 \leq t \leq \dfrac{5\pi}{2}$.

Solution: The total distance traveled is the integral of $|\cos t|$ over $[0, 5\pi/2]$.

$$\begin{aligned} \int_0^{5\pi/2} |\cos t| dt &= \int_0^{\pi/2} \cos t\, dt - \int_{\pi/2}^{3\pi/2} \cos t\, dt + \int_{3\pi/2}^{5\pi/2} \cos t\, dt \\ &= \sin t|_0^{\pi/2} - \sin t|_{\pi/2}^{3\pi/2} + \sin t|_{3\pi/2}^{5\pi/2} = 1 - 0 + 1 + 1 + 1 + 1 = 5 \end{aligned}$$

29. (6 points) Evaluate the integral. $\displaystyle\int x\sqrt{2+x}\, dx$

Solution: We use the substitution $u = 2 + x$. Then $du = dx$ and $x = u - 2$.

Then

$$\begin{aligned} \int x\sqrt{2+x}\, dx &= \int (u-2)u^{1/2}\, du = \int (u^{3/2} - 2u^{1/2})\, du \\ &= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C \\ &= \frac{2}{5}(2+x)^{5/2} - \frac{4}{3}(2+x)^{3/2} + C. \end{aligned}$$

END OF EXAM