Part I: Multiple Choice. Enter your answer on the scantron. Work will not be collected or reviewed.
21. (8 points) **Short answer.** Two points each part. You do not need to show your work on this problem.

(a) Given \( \varepsilon > 0 \), and the statement

\[
\lim_{x \to -1} (-2x + 3) = 5,
\]

find the largest \( \delta > 0 \) so that the following statement is true:

If \( 0 < |x + 1| < \delta \), then \( |-2x + 3 - 5| < \varepsilon \).

**Solution:** \( \varepsilon/2 \)

(b) Find \( \lim_{x \to \infty} \frac{5 - 3x^3}{\sqrt{81x^6 - 16}} \).

**Solution:** \(-1/3\)

(c) Evaluate the integral \( \int (1 + x^2) \, dx \).

**Solution:** \( x + x^3/3 + C \)

(d) Let \( f(x) \) be the function whose graph is shown below. Use right hand sums with four rectangles to estimate \( \int_0^2 f(x) \, dx \).

**Solution:** \( 11/2 = 5.5 \)

22. (5 points) Use the definition of the derivative to show:

If \( f(x) = \frac{1}{x - 1} \), then \( f'(x) = -\frac{1}{(x - 1)^2} \).
No credit will be given if a method besides the definition of the derivative is used.

Solution:

\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{x+h-1} - \frac{1}{x-1}}{h} \]

\[ = \lim_{h \to 0} \frac{x - 1 - x - h + 1}{h(x + h - 1)(x - 1)} = \lim_{h \to 0} \frac{-h}{h(x + h - 1)(x - 1)} \]

\[ = \lim_{h \to 0} \frac{-1}{(x+h-1)(x-1)} = -\frac{1}{(x-1)^2}. \]

23. (4 points) Find the linear approximation of the function \( f(x) = x^{3/2} \) at \( a = 100 \), and use it to approximate the number \( \sqrt{101}^3 \).

Solution: The formula for linear approximation is given by \( L(x) = f(a) + f'(a)(x - a) \).

First, \( f(a) = f(100) = (100)^{3/2} = 10^3 = 1000 \).

Next, \( f'(x) = \frac{3}{2}x^{1/2} \), so \( f'(a) = f'(100) = \frac{3}{2}(100)^{1/2} = 15 \).

Then the linear approximation of \( f(x) \) is given by

\[ L(x) = 1000 + 15(x - 100). \]

So \( \sqrt{101}^3 \approx 1000 + 15(101 - 100) = 1015 \).

24. (7 points) Prove that \( f(x) = 2x + \sin x \) has at most one root.

Solution: Suppose that \( f(x) \) has two roots, say there are numbers \( a \) and \( b \) such that \( f(a) = 0 = f(b) \). Since \( f \) is continuous and differentiable everywhere, it is continuous on \( [a, b] \) and differentiable on \( (a, b) \). Then we have the hypotheses of Rolles’ Theorem, so by Rolles’ Theorem there is a number \( c \) such that \( f'(c) = 0 \).

On the other hand, \( f'(x) = 2 + \cos x \). Since \(-1 \leq \cos x \leq 1\), \( 2 + \cos x \) can never equal 0. This is a contradiction.

Hence \( f(x) \) has at most one root.

25. (7 points) Find the limit. \( \lim_{x \to \infty} \left( \frac{1}{x^2} \right)^{1/x} \)

Solution: If we let \( L = \lim_{x \to \infty} \left( \frac{1}{x^2} \right)^{1/x} \), then

\[ \ln L = \lim_{x \to \infty} \ln \left( \frac{1}{x^2} \right)^{1/x} = \lim_{x \to \infty} \frac{\ln(1) - \ln(x^2)}{x} = \lim_{x \to \infty} \frac{-2 \ln(x)}{x} \]

Note that the limit on the far right is indeterminate of type \( \frac{\infty}{\infty} \), so we apply L’Hospital’s rule to obtain:

\[ \ln L = \lim_{x \to \infty} \frac{-2}{1} = 0 \]

So \( L = e^0 = 1 \).
26. (8 points) Suppose the area of a right triangle is 18 cm\(^2\). Find the smallest possible length of its hypoteneuse.

**Solution:** For a right triangle, let \( b \) denote the base and \( a \) the height. Then the area is given by

\[ A = \frac{1}{2}ab, \]

and the square of the hypoteneuse is given by:

\[ h^2 = a^2 + b^2. \]

We know that \( A = 18 \) (cm\(^2\)), hence \( a = 36/b \). Substitute this into our formula for \( h^2 \):

\[ h^2 = \frac{(36)^2}{b^2} + b^2. \]

Now, notice that \( h^2 \) and \( h \) will both have a minimum at the same values of \( a \) and \( b \), so we can save ourselves a little algebra and minimize \( H(b) = \frac{(36)^2}{b^2} + b^2 \) rather than its square root (although using the square root will give the same answer).

We find the derivative:

\[ H'(b) = -2 \frac{(36)^2}{b^3} + 2b \]

Set it equal to zero and solve for \( b \):

\[-2 \frac{(36)^2}{b^3} + 2b = 0 \iff -2(36)^2 + 2b^4 = 0 \iff b^4 = (36)^2 \iff b = 6 \text{ (cm)}\]

Use the second derivative test to check that this really gives a minimum:

\[ H''(b) = 6 \frac{(36)^2}{b^4} + 2 > 0 \text{ when } b = 6. \]

So \( b = 6 \) is a minimum, and \( H(6) = \frac{(36)^2}{36} + 36 = 72 \). Hence the shortest possible hypoteneuse is the square root of this, or \( 6\sqrt{2} \) cm.

27. (8 points) The graph of a function \( f(x) \) is shown. Let \( g(x) = \int_0^x f(t) \, dt \), for \( 0 \leq x \leq 5 \).
(a) Evaluate $g(0)$, $g(2)$, and $g(5)$.

**Solution:** $g(0) = 0, g(2) = 2.5, g(5) = -3.5$

(b) Where is $g(x)$ increasing on $[0, 5]$? Decreasing?

**Solution:** Increasing on $(0, 2)$, decreasing on $(2, 5)$.

(c) Sketch the graph of $g(x)$ on the same axes of $f(x)$, labeling all local maxima and minima.

**Solution:** Graph should include the three points $(0, 0)$, $(2, 2.5)$, and $(5, -3.5)$. It should be increasing from $(0, 0)$ to $(2, 2.5)$, then decreasing from $(2, 2.5)$ to $(5, -3.5)$. The point $(2, 2.5)$ is a local maximum. There are no local minima.

28. (7 points) A pendulum swings with velocity $v(t) = \cos t$. Find the total distance the pendulum travels over $0 \leq t \leq \frac{5\pi}{2}$.

**Solution:** The total distance traveled is the integral of $|\cos t|$ over $[0, \frac{5\pi}{2}]$.

\[
\int_0^{\frac{5\pi}{2}} |\cos t| \, dt = \int_0^{\frac{\pi}{2}} \cos t \, dt - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos t \, dt + \int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \cos t \, dt
\]

\[
= \sin t|_0^{\frac{\pi}{2}} - \sin t|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} + \sin t|_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} = 1 - 0 + 1 + 1 + 1 = 5
\]

29. (6 points) Evaluate the integral. $\int x\sqrt{2 + x} \, dx$

**Solution:** We use the substitution $u = 2 + x$. Then $du = dx$ and $x = u - 2$.

Then

\[
\int x\sqrt{2 + x} \, dx = \int (u - 2)u^{1/2} \, du = \int (u^{3/2} - 2u^{1/2}) \, du
\]

\[
= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C
\]

\[
= \frac{2}{5}(2 + x)^{5/2} - \frac{4}{3}(2 + x)^{3/2} + C.
\]