Math 112 (Calculus I)
Final Exam Form A KEY

Free response: Write your answer in the space provided. Answers not placed in this space will be ignored.

21. (8 points) Compute the integrals.

(a) $\displaystyle\int_0^1 \frac{e^{2t}+1}{e^{2t}+2t}\,dt$

Solution: Let $u = e^{2t}+2t$. Then $du = (2e^{2t}+2)\,dt = \frac{1}{2}(e^{2t}+1)\,dt$.

Also, when $t=0$, $u = e^0+0 = 1$, and when $t=1$, $u = e^2+2$. Hence by substitution, the integral becomes:

$$\int_{u=1}^{e^2+2} \frac{du}{2u} = \frac{1}{2}\ln(u)\Big|_{u=1}^{e^2+2} = \frac{1}{2}\ln(e^2+2) - \frac{1}{2}\ln(1) = \frac{1}{2}\ln(e^2+2).$$

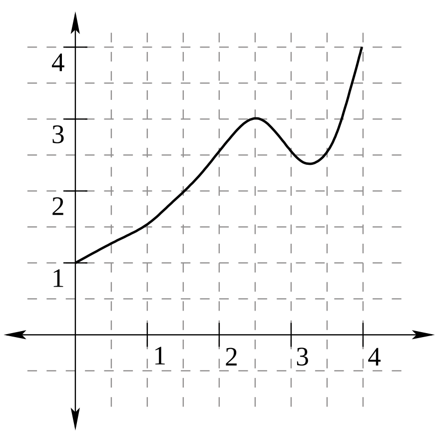
(b) $\displaystyle\int_1^2 \frac{x^3+6x^6}{x^5}\,dx$

Solution: $\displaystyle\int_1^2 \frac{x^3+6x^6}{x^5}\,dx = \int_1^2 (\frac{1}{x^2}+6x)\,dx = (-\frac{1}{x}+3x^2)\Big|_1^2 = (-\frac{1}{2}+12)-(-1+3) =$ 19/2 or 9.5

22. (6 points) The graph of the function f is shown below. Estimate $\displaystyle\int_1^3 f(x)\,dx$ in two ways:

(a) Using four sub-intervals and right endpoints.

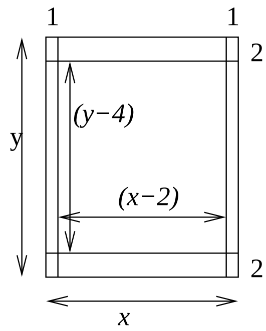
(b) Using four sub-intervals and left endpoints.



Solution: (a) $\frac{1}{2}(2+\frac{5}{2}+3+\frac{5}{2}) = \frac{20}{4} = 5$

(b) $\frac{1}{2}(\frac{3}{2}+2+\frac{5}{2}+3) = \frac{18}{4} = \frac{9}{2}$

23. (8 points) A printed poster is to have area 200 square inches with 2 inch margins on the top and bottom, and 1 inch margins on the sides. What dimensions for the poster will ensure that the printed area is as large as possible?



Solution:

Let x and y denote the side lengths, as in the figure. Then $xy = 200$ and the printed area that we want to maximize is $A = (x-2)(y-4)$.

Solve for y in the first equation: $y = 200/x$.

The second equation becomes: $A = (x-2)(200/x - 4) = 200 - 4x - 400/x + 8$.

Take the derivative of $A(x)$ and set it equal to zero: $-4 + 400/x^2 = 0$ or $x^2 = 100$, so $x = 10$.

Check $x = 10$ is indeed a maximum: $A''(x) = -800/x^3$, which is negative when $x = 10$, so the second derivative test implies $x = 10$ is a maximum.

Then $y = 200/x = 200/10 = 20$.

Thus the printed area is as large as possible when the sides are 10 inches by 20 inches.

24. (12 points) Compute the following limits. If the limit is infinite, explain whether the limit is ∞ or $-\infty$ or neither.

(a) $\lim_{x\to 0} \frac{x+1-e^x}{5x^2}$

Solution: This has an indeterminate form of $\frac{0}{0}$, so L'Hospital's rule implies the limit equals: $\lim_{x\to 0} \frac{1-e^x}{10x}$.

Again this is type $\frac{0}{0}$, so apply L'Hospital's rule again to find the limit equals $\lim_{x\to 0} \frac{-e^x}{10} = -\frac{1}{10}$.

(b) $\lim_{x\to 3} \frac{4x}{(x-3)^2}$

Solution: As $x \to 3$, the numerator approaches 12, but the denominator approaches 0. Thus the funciton $\frac{4x}{(x-3)^2}$ has an asymptote at $x = 3$. We check whether the limit can be described as positive or negative infinity, or whether it doesn't exist.

As $x \to 3$ from the right or left, $(x-3)^2 > 0$. Hence the limit approaches positive infinity.

(c) $\lim_{x\to 0} 3x^2 \sin\left(\frac{1}{x^2}\right)$

Solution: Note that:

$$-1 \le \sin\left(\frac{1}{x^2}\right) \le 1, \text{ so}$$

$$-3x^2 \le 3x^2 \sin\left(\frac{1}{x^2}\right) \le 3x^2.$$

Since $\lim_{x\to 0}(-3x^2) = 0 = \lim_{x\to 0} 3x^2$, by the Squeeze theorem, the limit is 0.

25. (5 points) Use implicit differentiation along with the definition of the function $y = \ln(x)$ as the inverse of the exponential function to prove that

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}.$$

No credit will be given if a method other than implicit differentiation is used.

Solution: Let $y = \ln x$. Then $e^y = x$

Differentiate both sides:

$$\frac{d}{dx}e^y = \frac{d}{dx}x$$
$$e^y\frac{dy}{dx} = 1$$

Hence $\frac{dy}{dx} = \frac{1}{e^y}$. Now substitute back $e^y = x$:

$$\frac{dy}{dx} = \frac{1}{x}.$$

26. (6 points) Prove that the function $f(x) = -x - 3^x$ has at least one root in the interval $(-1, 1)$.

Solution: Note that $f(x)$ is continuous on $[0, 1]$.

Note also that $f(0) = 1 - 0 - 0 = 1 > 0$,
and $f(1) = 1 - 1 - 1 = -1 < 0$.

Then the intermediate value theorem implies there exists $c \in (0, 1)$ such that $f(c) = 0$.

27. (8 points) Gravel is being poured onto the top of a pile that forms a right circular cone in such a way that the radius of the cone is always three times its height. If the gravel is being poured at a rate of 18 m^3/min, at what rate is the height of the cone changing when the height is 2 m? ($V = \frac{1}{3}\pi r^2 h$.)

Solution: We know that $r = 3h$, and $\frac{dV}{dt} = 18\ \text{m}^3/\text{min}$.

So $V = \frac{1}{3}\pi(3h)^2 h = 3\pi h^3$.

Thus $\frac{dV}{dt} = 9\pi h^2 \frac{dh}{dt}$.
When $h = 2$ m:

$$18 = 9\pi(4)\frac{dh}{dt} \text{ hence } \frac{dh}{dt} = \frac{18}{36\pi} = \frac{1}{2\pi}\ \text{m/min}.$$

28. (7 points) Determine where the following function is concave up and concave down on the interval $(0, 2\pi)$.

$$f(x) = e^x \sin(x)$$

Solution:

$$f'(x) = e^x \sin x + e^x \cos x$$
$$f''(x) = e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x = 2e^x \cos x.$$

Now $2e^x \cos x = 0$ if and only if $\cos x = 0$, and $\cos x = 0$ at $x = \pi/2$ and $x = 3\pi/2$ in the interval $(0, 2\pi)$.

Check: $2e^x \cos x$ is positive in $(0, \pi/2)$, negative in $(\pi/2, 3\pi/2)$, and positive in $(3\pi/2, 2\pi)$.

Hence f is concave up on $(0, \pi/2) \cup (3\pi/2, 2\pi)$ and concave down on $(\pi/2, 3\pi/2)$.

END OF EXAM