Math 112 – Winter 2007 — Key

Departmental Final Exam

Part I: Fill in the blank or circle T/F

1. (a) The limit $\lim_{x\to\infty} \frac{2x^2+3x}{1-x^2} = \boxed{-2}$

(b) If $f(x) = x^2$, and $x_0 = 1$, then Newton's method for solving $f(x) = 0$ gives us $x_1 = \boxed{\frac{1}{2}}$

(c) ($\boxed{\text{T}}$ / F) If $f''(x)$ exists on $[a, b]$, then $f(x)$ is continuous on $[a, b]$

(d) The mean value theorem states that if f is differentiable on $[a, b]$, then there is a c in (a, b) with

$$f'(c) = \boxed{\frac{f(b) - f(a)}{b - a}}$$

(e) The limit $\lim_{x\to 3+} \frac{|x-3|}{x-3} = \boxed{1}$

(f) The average value of a function f over an interval $[a, b]$ is given by $\boxed{\frac{1}{b-a}\int_a^b f(x)\,dx}$

(g) If $\int_2^4 f(x)\,dx = 2$, $\int_0^4 f(x)\,dx = 6$, $\int_0^2 g(x)\,dx = 5$, then

$$\int_0^2 f(x) + 3g(x)\,dx = \boxed{19}$$

(h) ($\boxed{\text{T}}$/F) If $f'(x)$ exists on $[a, b]$, then $f(x)$ is integrable on $[a, b]$.

(i) The integral $\int \frac{dx}{1+x^2} = \boxed{\tan^{-1} x + C}$

Part II: Multiple Choice

Problems 2 through 8 are multiple choice. Each multiple choice problem is worth 4 points. In the grid below fill in the square corresponding to each correct answer.

2	A	■	C	D	E	F	G	H	I	J
3	A	B	C	■	E	F	G	H	I	J
4	A	B	C	D	E	F	■	H	I	J
5	A	B	C	D	■	F	G	H	I	J
6	A	B	C	D	■	F	G	H	I	J
7	A	B	C	D	E	■	G	H	I	J
8	A	B	C	D	■	F	G	H	I	J

2. $\lim_{x\to 0} \dfrac{2x}{\sin 5x} =$

   A. 0 B. $\boxed{\dfrac{2}{5}}$ C. 1 D. $\dfrac{5}{2}$
   E. 2 F. $-\infty$ G. ∞ H. Limit does not exist.

3. Given the limit statement $\lim_{x\to 5}(-3x+17) = 2$, pick the largest δ that works with the definition of the limit if $\epsilon = 0.06$.

   A. 0.001 B. 0.005 C. 0.01 D. $\boxed{0.02}$
   E. 0.03 F. 0.06 G. 0 H. No such δ.

4. If $f(x) = 6x^2,\ g(-1) = -2,\ g'(-1) = 3$, find $\dfrac{d}{dx}\left(f(g(x))\right)$ at $x = -1$.

   A. 0 B. 1 C. -12 D. -24
   E. -36 F. -48 G. $\boxed{-72}$ H. None of the above.

5. Which of the following is the maximum value of $f(x) = 2x^3 - 3x^2 - 36x + 4$ over $[-3, 2]$?

   A. 4 B. -2 C. 3 D. 80
   E. $\boxed{48}$ F. 64 G. -16 H. None of the above.

6. Given $x^2 \ln y + y \ln(x^2) = 2e$, find $\dfrac{dy}{dx}$ at the point $(\sqrt{e}, e)$.

   A. -1 B. e C. $-2e$ D. $\dfrac{\sqrt{e}}{2\sqrt{e}-2}$
   E. $\boxed{-2\sqrt{e}}$ F. $-\sqrt{2e}$ G. 1 H. $\dfrac{2\sqrt{e}+2}{\sqrt{e}-2}$

7. Which of the following is a Riemann sum for $\int_0^1 \sinh^{-1} x\, dx$ as $n \to \infty$?

A. $\displaystyle\sum_{j=0}^{n-1} \sinh^{-1}\left(\frac{1}{n}\right) \cdot \frac{j}{n}$ B. $\displaystyle\sum_{j=0}^{n-1} \sinh^{-1}\left(\frac{j+1}{n}\right) \cdot \frac{1}{n}$ C. $\displaystyle\sum_{j=0}^{n-1} \sinh^{-1}\left(\frac{j}{n}\right) \cdot \frac{j}{n}$

D. $\displaystyle\sum_{j=0}^{n-1} \sinh^{-1}\left(\frac{2j+1}{2n}\right) \cdot \frac{1}{n}$ E. *(A) and (C)* F. $\boxed{\textit{(B) and (D)}}$

G. All of the above H. None of the above

8. Evaluate $\displaystyle\int_0^{1/2} 8(1-4x)^3\, dx$

A. -1 B. 1 C. 4 D. -4

E. $\boxed{0}$ F. 3 G. -5 H. None of the above.

The answers to the multiple choice MUST be entered on the grid on page 2. Otherwise, you will not receive credit.

Part III: Written Solutions

For problems 9 - 18, write your answers in the space provided. Neatly show your work for full credit.

9. (a) State the conditions for $f(x)$ defined over $[0, 2]$ to be continuous at $x = 1$.

   i. $f(x)$ is defined at $x = 1$
   ii. $\lim_{x \to 1} f(x)$ exists and is equal to $f(1)$.

   (b) At which points does the function

$$f(x) = \frac{\sqrt{x+4}}{(x+2)(x-3)}$$

   fail to be continuous? At which points, if any, are the discontinuities removable? not removable? Give reasons for your answers.

The function f is not defined for all $x < -4$ and at $x = -2, 3$ and thus discontinuous at these points.

All the discontinuous points are not removable because at these points the limit of f does not exist.

10. Differentiate the following:

(a) $f(x) = x^\pi + \sec(\tan x)$ Applying power rule and chain rule, and recalling that

$$(\sec x)' = \sec x \, \tan x, \quad (\tan x)' = \sec^2 x,$$

$$f'(x) = \pi \, x^{\pi - 1} + \sec(\tan x) \, \tan(\tan x) \, \sec^2 x$$

(b) $g(x) = \left(\dfrac{x^2+1}{x^2-1}\right)^{1/2}$

Applying chain rule and quotient rule

$$\begin{aligned} g'(x) &= \frac{1}{2}\left(\frac{x^2+1}{x^2-1}\right)^{-1/2}\left(\frac{x^2+1}{x^2-1}\right)' = \frac{1}{2}\left(\frac{x^2+1}{x^2-1}\right)^{-1/2}\frac{-4x}{(x^2-1)^2} \\ &= \left(\frac{x^2+1}{x^2-1}\right)^{-1/2}\cdot\frac{-2x}{(x^2-1)^2} \end{aligned}$$

11. Find the equation for the tangent line to $f(x) = e^x \cos(x)$ at the point $(\pi, -e^\pi)$.

$$f'(x) = e^x \cos(x) - e^x \sin(x) \quad \Rightarrow \quad f'(\pi) = -e^\pi$$

so the equation for the tangent line at the point $(\pi, -e^\pi)$ is given by

$$y + e^\pi = -e^\pi(x - \pi)$$

or

$$y = -e^\pi x + (\pi - 1)e^\pi$$

12. If $f(x) = \dfrac{1}{x-1}$, find $f'(2)$ using the definition of the derivative. (*No point will be awarded if differentiation rules are used.*)

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\frac{1}{2+h-1} - \frac{1}{2-1}}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{1+h} - 1\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{1-(1+h)}{1+h}\right) = \lim_{h\to 0} \frac{-1}{1+h} \\ &= -1 \end{aligned}$$

Alternatively,

$$\begin{aligned} f'(2) &= \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2} \frac{\frac{1}{x-1} - \frac{1}{2-1}}{x-2} \\ &= \lim_{x\to 2} \frac{1}{x-2}\left(\frac{1}{x-1} - 1\right) = \lim_{x\to 2} \frac{1}{x-2}\left(\frac{1-(x-1)}{x-1}\right) = \lim_{x\to 2} \frac{-1}{x-1} \\ &= -1 \end{aligned}$$

13. What are the dimensions of the rectangle of largest area that fits in a right triangle with side lengths 3 in, 4 in and 5 in?

Suppose the vertices of the triangle are located at the origin, $(0, 4)$ and $(3, 0)$. Let x and y be the width and the height of a rectangle with sides parallel to the axes and a corner lying on the hypothenuse. Then

$$\frac{x}{3} + \frac{y}{4} = 1 \quad \Rightarrow \quad y = 4\left(1 - \frac{x}{3}\right)$$

thus the area of the rectangle is

$$A = xy = 4x\left(1 - \frac{x}{3}\right)$$

$$A'(x) = 4 - \frac{8x}{3}$$

Setting the derivative to 0 and solving, we have

$$x = \frac{3}{2}, \quad y = 2$$

Note the area is largest from the second derivative test with $A'' = -\frac{8}{3} < 0$

14. The area of a square is increasing at 4 in^2/s. How fast is the length of the diagonal increasing at the moment that the side of the square is 6 in?

Let x the side of the square. Then the rate of increase of the area $A = x^2$ is given by $\dfrac{dA}{dt} = 4$ in^2/s. Now

$$\frac{dA}{dt} = 2x\,\frac{dx}{dt} \qquad \Rightarrow \qquad \frac{dx}{dt} = \frac{2}{x}$$

The length of the diagonal is $\ell(x) = \sqrt{2}\,x$, so the rate of increase of the diagonal is

$$\frac{d\ell}{dt} = \sqrt{2}\,\frac{dx}{dt} = \frac{2\sqrt{2}}{x}$$

Hence when the side of the square is 6 in, the diagonal is increasing at a rate of

$$\frac{d\ell}{dt} = \frac{\sqrt{2}}{3} \qquad \text{(in/s)}$$

15. Given that for all $x > -4$, the function $f(x)$ is defined, continuous and satisfies the bounds

$$\frac{2}{1 + e^{-1/x^2}} \leq f(x) \leq 2 + \frac{x}{4 - \sqrt{x+4}}$$

Determine the value $f(0)$. State any theorem you used to find your answer.

As

$$\lim_{x \to 0} \frac{2}{1 + e^{-1/x^2}} = \frac{2}{1+0} = 2$$

and

$$\lim_{x \to 0} 2 + \frac{x}{4 - \sqrt{x+4}} = \lim_{x \to 0} 2 + \frac{x(4 + \sqrt{x+4})}{16 - (x+4)} = 2$$

so by the squeeze play or comparison limit theorem,

$$\lim_{x \to 0} f(x) = 2.$$

Since $f(x)$ is continuous , so $f(0) = 2$.

16. Let $A(x) = \int_{x+1}^{\sqrt{x}} \sin t^2\, dt$. Find $\dfrac{dA}{dx}$. State any theorem you used to find your answer.

By the fundamental theorem of calculus and the chain rule,

$$\frac{dA}{dx} = \frac{d}{dx}\int_{x+1}^{\sqrt{x}} \sin t^2\, dt = \sin\left((\sqrt{x})^2\right)\,\frac{d}{dx}(\sqrt{x}) \;-\; \sin\left((x+1)^2\right)\,\frac{d}{dx}(x+1)$$

and so

$$\frac{dA}{dx} = \frac{\sin x}{2\sqrt{x}} - \sin\left((x+1)^2\right)$$

17. If

$$f(x) = \frac{x}{x^2+1},$$

find all intervals of monotonicity, all intervals of concavity, all inflection points, all relative extrema and all global extrema if possible.

$$f' = \frac{1-x^2}{(x^2+1)^2}, \qquad f'' = \frac{2x(x^2-3)}{(x^2+1)^3}$$

so inflection points are $x = 0, \pm\sqrt{3}$ and extrema are $x = \pm 1$.

x	$f''(x)$	$f'(x)$	$f(x)$
-1	$\frac{1}{2}$	0	relative min
1	$-\frac{1}{2}$	0	relative max
< -1		$-$	decreasing
$(-1, 1)$		$+$	increasing
> 1		$-$	decreasing
$< -\sqrt{3}$	$-$		conave down
$(-\sqrt{3}, 0)$	$+$		conave up
$(0, \sqrt{3})$	$-$		conave down
$> \sqrt{3}$	$+$		conave up

The only relative maximum is also the global maximum. The only relative minimum is also the global minimum.

18. Find the following integrals

(a) $\displaystyle\int_{e^2}^{e^3} 2x^{-1}\,dx$

$$\begin{aligned}\int_{e^2}^{e^3} 2x^{-1}\,dx &= 2\ln x|_{e^2}^{e^3} = 2\ln e^3 - 2\ln e^2 \\ &= 6-4=2\end{aligned}$$

(b) $\displaystyle\int \frac{\cos^4(\sqrt{x})\sin(\sqrt{x})}{\sqrt{x}}\,dx$

Using substitution $u=\cos(\sqrt{x})$,

$$du = -\sin(\sqrt{x})\,\frac{1}{2\sqrt{x}}\,dx$$

$$\begin{aligned}\int \frac{\cos^4(\sqrt{x})\sin(\sqrt{x})}{\sqrt{x}}\,dx &= \int u^4(-2)\,du = -\frac{2}{5}u^5 + C \\ &= -\frac{2}{5}\cos^4(\sqrt{x}) + C\end{aligned}$$

(c) $\displaystyle\int_0^1 30x\sqrt{1-x}\,dx$

Using substitution $u^2 = 1-x$,

$$2udu = -dx, \qquad x = 0,1 \Rightarrow u = 1,0 \text{ resp.}$$

$$\begin{aligned}\int_0^1 30x\sqrt{1-x}\,dx &= \int_1^0 30(1-u^2)u\,(-2u)\,du = \int_0^1 60(u^2-u^4)\,du \\ &= 20u^3 - 12u^5\Big|_0^1 = 8\end{aligned}$$

—*End*—