Math 112 (Calculus I)
Final Exam Form A KEY

Multiple Choice. Fill in the answer to each problem on your scantron. Make sure your name, section and instructor is on your scantron.

1. Find the absolute *minimum* value for $f(x)$ on the interval $[-4, 3]$ when $f(x)$ is given by

$$f(x) = \frac{x^2 - 4}{x^2 + 4}.$$

a) -1	b) $5/13$	c) $4/5$
d) -2	e) 2	f) 1

Solution: a)

2. For what value of c is the function $g(x)$ below continuous?

$$g(x) = \begin{cases} \dfrac{cx^2 - 4c}{x - 2} & \text{if } x < 2 \\ cx + 1 & \text{if } x \geq 2 \end{cases}$$

a) $c = 1$	b) $c = 2$	c) $c = 1/2$
d) $c = -2$	e) $c = 0$	f) None of the above

Solution: c)

3. Evaluate:

$$\frac{d}{dx} \int_3^{x^2} \ln(t - 1)\, dt$$

a) $2x \ln(x^2 - 1)$	b) $\dfrac{1}{x^2 - 1} - \dfrac{1}{8}$	c) $\ln(x^2 - 1) - \ln(8)$
d) $2x \ln(x^2 - 1) - 2x \ln(8)$	e) $2x \ln((x^2 - 1)^4 (x^2 - 1))$	f) $\dfrac{2x}{x^2 - 1} + \dfrac{x}{4}$

Solution: a)

4. Use linear approximation to estimate $\sinh(0.1)$.

a) 0.2 b) 0.01 c) -0.2

d) 1.1 e) 0.9 f) 0.1

Solution: f)

5. A function $s(t)$ is given by

$$s(t) = t^{1/3} + t^{2/3}.$$

Find $s''(1)$.

a) 1 b) $-1/3$ c) 0

d) $-4/9$ e) $1/3$ f) -1

Solution: d)

6. Find all vertical asymptotes.

$$y = \frac{3(x-2)(x+1)\ln(x+5)}{(x+4)(x-2)}$$

i. $x = -4$ ii. $x = 2$ iii. $x = -5$ iv. $x = 1$

a) i. only b) i. and ii. c) i., ii., and iii.

d) i., ii., iii., and iv. e) i. and iii. only f) ii. and iii. only

Solution: e)

7. For the definition of the limit

$$\lim_{x \to 2} 3x = 6,$$

find the largest value of δ that corresponds to $\epsilon = 0.06$.

a) 0.06 b) 0.02 c) 0.2

d) 0.1 e) 0.03 f) 0.01

Solution: b)

8. A certain isotope has a half-life of 40 years. Suppose we have a 100 mg sample. How much remains after 60 years?

a) $100/\sqrt[3]{4}$ mg b) 50 mg c) $100e^{3/2}$ mg

d) $25\sqrt{2}$ mg e) $200/3$ mg f) $100\ln(1/2)$ mg

Solution: d)

Short Answer. Fill in the blank with the appropriate answer.

9. A rectangular box with a square base is to hold 4000 cubic centimeters. Material for the sides costs 2 cents per square centimeter, while material for the top and bottom costs 1 cent per square centimeter.

   (a) (4 points) If the base has side x, express the cost C of the box as a function of x.

   Solution:

   Form A

$$x^2y = 4000, \quad y = \frac{4000}{x^2}$$

$$C = 2 \cdot 4xy + 1 \cdot 2x^2 = \frac{32000}{x} + 2x^2.$$

   Form B

$$x^2y = 9000, \quad y = \frac{9000}{x^2}$$

$$C = 6 \cdot 4xy + 2 \cdot 2x^2 = \frac{216000}{x} + 4x^2.$$

   (b) (4 points) Find the dimensions of the most economical box.

   Solution:

   Form A

$$C' = -\frac{32000}{x^2} + 4x = \frac{4x^3 - 32000}{x^2}.$$

   Thus,

$$x^3 = \frac{32000}{4} = 8000, x = 20, y = 10.$$

   Form B

$$C' = -\frac{216000}{x^2} + 8x = \frac{8x^3 - 216000}{x^2}.$$

   Thus,

$$x^3 = \frac{216000}{8} = 27000, x = 30, y = 10$$

10. (7 points) Find the equation of the tangent line to the curve given implicitly by

$$e^{xy} = x^2 + y^2$$

at the point $(0, 1)$.

Solution: By differentiating implicitly, we have

$$e^{xy}(y + xy') = 2x + 2yy'$$

Plugging in $x = 0$ and $y = 1$, we have

$$e^0(1) = 2y', \ y' = \frac{1}{2}$$

11. (7 points) A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 2 m/s, at what rate is the angle between the rope and the horizontal increasing when that angle is $\pi/6$?

Solution:

Form A If s is the length of the rope and θ is the angle the rope makes from the horizontal,

$$s \sin\theta = 1,$$

and

$$\sin\theta \frac{ds}{dt} + s\cos\theta \frac{d\theta}{dt} = 0.$$

When $\theta = \pi/6$, $s = 2$. Thus,

$$-\frac{1}{2} \cdot 2 + 2 \cdot \frac{\sqrt{3}}{2} \frac{d\theta}{dt} = 0,$$

or

$$\frac{d\theta}{dt} = \frac{1}{\sqrt{3}}.$$

Form B When $\theta = \pi/4$, $s = \sqrt{2}$. Thus,

$$-\frac{\sqrt{2}}{2} \cdot 3 + \sqrt{2} \cdot \frac{\sqrt{2}}{2} \frac{d\theta}{dt} = 0,$$

or

$$\frac{d\theta}{dt} = \frac{3}{\sqrt{2}}.$$

12. (7 points) Find the derivative of $(\arctan(x))^{3x}$. *Don't simplify your answer.*

Solution:

Form A

$$\ln y = 3x \ln(\arctan(x))$$

$$\frac{1}{y} y' = 3\ln(\arctan(x)) + \frac{3x}{\arctan(x)(1+x^2)}$$

$$y' = (\arctan(x))^{3x} \left(3\ln(\arctan(x)) + \frac{3x}{\arctan(x)(1+x^2)} \right)$$

Form B

$$y' = (\arctan(x))^{4x} \left(4\ln(\arctan(x)) + \frac{4x}{\arctan(x)(1+x^2)} \right)$$

13. Evaluate the following limits.

(a) (6 points) $\lim_{x\to\infty} 3x^2\, e^{-4x}$

Solution:

Form A

$$\lim_{x\to\infty} 3x^2\, e^{-4x} = \lim_{x\to\infty} \frac{3x^2}{e^{4x}} = \lim_{x\to\infty} \frac{6x}{4e^{4x}}$$
$$= \lim_{x\to\infty} \frac{6}{16e^{4x}} = 0.$$

Form B

$$\lim_{x\to\infty} 2x^2\, e^{-6x} = \lim_{x\to\infty} \frac{2x^2}{e^{5x}} = \lim_{x\to\infty} \frac{4x}{5e^{5x}}$$
$$= \lim_{x\to\infty} \frac{4}{25e^{5x}} = 0.$$

(b) (6 points) $\lim_{h\to 0} \frac{a^{1+h} - a}{h}$

Solution:

$$\lim_{h\to 0} \frac{a^{1+h} - a}{h} = \lim_{h\to 0} \frac{\ln(a)a^{1+h}}{1} = \ln(a)a.$$

14. (6 points) Write $\lim_{n\to\infty} \sum_{i=1}^{n} e^{i/n} \left(\frac{1}{n}\right)$ as a definite integral and calculate it.

Solution:

$$\int_0^1 e^x\, dx = e^x\big|_0^1 = e - 1.$$

15. (6 points) Use the definition of the derivative to find $f'(x)$ when $f(x) = \sqrt{x}$.

Solution:

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

16. (8 points) Show $\ln(x) + x - 2 = 0$ has exactly one real root in the interval $(1, e)$.

Solution: If $f(x) = \ln(x) + x - 2$, then $f(1) = -1 < 0$ and $f(e) = e - 1 > 0$.Thus, there is at least one root by the intermediate value theorem.

If f had two roots, then Rolle's theorem would tell us that $f'(c) = 0$ somewhere between. However,

$$f'(x) = \frac{1}{x} + 1$$

is positive on the entire interval. Hence, there is only one root.

17. Evaluate the integrals.

(a) (5 points) $\int_0^1 v^2 \cos(v^3)\, dv$

Solution: $u = v^3$, $du = 3v^2\, dv$.

$$\int_0^1 v^2 \cos(v^3)\, dv = \int_0^1 \cos(u) \frac{du}{3} = \frac{1}{3}(\sin(1)).$$

(b) (5 points) $\int \frac{\sec\theta \tan\theta}{1+\sec\theta}\, d\theta$

Solution:

Form A $u = 1 + \sec\theta$, $du = \sec\theta \tan\theta\, d\theta$.

$$\int \frac{\sec\theta \tan\theta}{1+\sec\theta}\, d\theta = \int \frac{du}{u} = \ln|u| + C$$
$$= \ln|1+\sec\theta| + C.$$

Form B

$$\ln|3+\sec\theta| + C.$$

(c) (5 points) $\int_0^3 |x^2 - 4|\, dx$

Solution:

$$\int_0^3 |x^2-4|\, dx = \int_0^2 4 - x^2\, dx + \int_2^3 x^2 - 4\, dx$$
$$= (4x - \frac{x^3}{3})|_0^2 + (\frac{x^3}{2} - 4x)|_2^4 = 8 - \frac{8}{3} + \frac{64}{3} - 16 - (\frac{8}{3} - 8)$$
$$= \frac{48}{3}$$

END OF EXAM