Math 112 (Calculus I)
Final Exam Form A KEY

Multiple Choice. Fill in the answer to each problem on your scantron. Make sure your name, section and instructor is on your scantron.

1. Find the absolute minimum value for \( f(x) \) on the interval \([-4, 3]\) when \( f(x) \) is given by

\[
f(x) = \frac{x^2 - 4}{x^2 + 4}.
\]

a) \(-1\) b) \(5/13\) c) \(4/5\)

\[\text{d) } -2 \quad \text{e) } 2 \quad \text{f) } 1\]

Solution: a)

2. For what value of \( c \) is the function \( g(x) \) below continuous?

\[
g(x) = \begin{cases} 
\frac{cx^2 - 4c}{x - 2} & \text{if } x < 2 \\
\frac{cx + 1}{x - 2} & \text{if } x \geq 2 
\end{cases}
\]

a) \(c = 1\) b) \(c = 2\) c) \(c = 1/2\)

\[\text{d) } c = -2 \quad \text{e) } c = 0 \quad \text{f) None of the above}\]

Solution: c)

3. Evaluate:

\[
\frac{d}{dx} \int_3^{x^2} \ln(t - 1) \, dt
\]

a) \(2x \ln(x^2 - 1)\) b) \(\frac{1}{x^2 - 1} - \frac{1}{8}\) c) \(\ln(x^2 - 1) - \ln(8)\)

\[\text{d) } 2x \ln(x^2 - 1) - 2x \ln(8) \quad \text{e) } 2x \ln((x^2 - 1)^4(x^2 - 1)) \quad \text{f) } \frac{2x}{x^2 - 1} + \frac{x}{4}\]

Solution: a)
4. Use linear approximation to estimate sinh(0.1).

a) 0.2  

b) 0.01  

c) −0.2  

d) 1.1  

e) 0.9  

f) 0.1  

Solution: f)

5. A function \( s(t) \) is given by \( s(t) = t^{1/3} + t^{2/3}. \)

Find \( s''(1) \).

a) 1  

b) −1/3  

c) 0  

d) −4/9  

e) 1/3  

f) −1  

Solution: d)

6. Find all vertical asymptotes.

\[
y = \frac{3(x - 2)(x + 1)\ln(x + 5)}{(x + 4)(x - 2)}
\]

i. \( x = -4 \)  

ii. \( x = 2 \)  

iii. \( x = -5 \)  

iv. \( x = 1 \)

a) i. only  

b) i. and ii.  

c) i., ii., and iii.  

d) i., ii., iii., and iv.  

e) i. and iii. only  

f) ii. and iii. only  

Solution: e)

7. For the definition of the limit

\[
\lim_{x \to 2} 3x = 6,
\]

find the largest value of \( \delta \) that corresponds to \( \epsilon = 0.06 \).

a) 0.06  

b) 0.02  

c) 0.2  

d) 0.1  

e) 0.03  

f) 0.01  

Solution: b)
8. A certain isotope has a half-life of 40 years. Suppose we have a 100 mg sample. How much remains after 60 years?

   a) $\frac{100}{\sqrt[4]{4}}$ mg  
   b) 50 mg  
   c) $100e^{3/2}$ mg  
   d) $25\sqrt{2}$ mg  
   e) $\frac{200}{3}$ mg  
   f) $100 \ln(1/2)$ mg

Solution: d)
9. A rectangular box with a square base is to hold 4000 cubic centimeters. Material for the sides costs 2 cents per square centimeter, while material for the top and bottom costs 1 cent per square centimeter.

(a) (4 points) If the base has side $x$, express the cost $C$ of the box as a function of $x$.

Solution:

Form A

$x^2 y = 4000, \quad y = \frac{4000}{x^2}$

$C = 2 \cdot 4xy + 1 \cdot 2x^2 = \frac{32000}{x} + 2x^2$.

Form B

$x^2 y = 9000, \quad y = \frac{9000}{x^2}$

$C = 6 \cdot 4xy + 2 \cdot 2x^2 = \frac{216000}{x} + 4x^2$.

(b) (4 points) Find the dimensions of the most economical box.

Solution:

Form A

$C' = -\frac{32000}{x^2} + 4x = \frac{4x^3 - 32000}{x^2}$.

Thus,

$x^3 = \frac{32000}{4} = 8000, \quad x = 20, \quad y = 10$.

Form B

$C' = -\frac{216000}{x^2} + 8x = \frac{8x^3 - 216000}{x^2}$.

Thus,

$x^3 = \frac{216000}{8} = 27000, \quad x = 30, \quad y = 10$

10. (7 points) Find the equation of the tangent line to the curve given implicitly by

$e^{xy} = x^2 + y^2$

at the point $(0, 1)$.

Solution: By differentiating implicitly, we have

$e^{xy}(y + xy') = 2x + 2yy'$

Plugging in $x = 0$ and $y = 1$, we have

$e^0(1) = 2y', \quad y' = \frac{1}{2}$
11. (7 points) A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 2 m/s, at what rate is the angle between the rope and the horizontal increasing when that angle is π/6?

**Solution:**

**Form A** If $s$ is the length of the rope and $\theta$ is the angle the rope makes from the horizontal, 

$$ s \sin \theta = 1, $$

and

$$ \sin \theta \frac{ds}{dt} + s \cos \theta \frac{d\theta}{dt} = 0. $$

When $\theta = \pi/6$, $s = 2$. Thus,

$$ -\frac{1}{2} \cdot 2 + 2 \cdot \frac{\sqrt{3}}{2} \frac{d\theta}{dt} = 0, $$

or

$$ \frac{d\theta}{dt} = \frac{1}{\sqrt{3}}. $$

**Form B** When $\theta = \pi/4$, $s = \sqrt{2}$. Thus,

$$ -\frac{\sqrt{2}}{2} \cdot 3 + \sqrt{2} \cdot \frac{\sqrt{2}}{2} \frac{d\theta}{dt} = 0, $$

or

$$ \frac{d\theta}{dt} = \frac{3}{\sqrt{2}}. $$

12. (7 points) Find the derivative of $(\arctan(x))^{3x}$. *Don’t simplify your answer.*

**Solution:**

**Form A**

$$ \ln y = 3x \ln(\arctan(x)) $$

$$ \frac{1}{y'} = 3 \ln(\arctan(x)) + \frac{3x}{\arctan(x)(1+x^2)} $$

$$ y' = (\arctan(x))^{3x} \left( 3 \ln(\arctan(x)) + \frac{3x}{\arctan(x)(1+x^2)} \right) $$

**Form B**

$$ y' = (\arctan(x))^{4x} \left( 4 \ln(\arctan(x)) + \frac{4x}{\arctan(x)(1+x^2)} \right) $$
13. Evaluate the following limits.

(a) (6 points) \( \lim_{x \to \infty} 3x^2 e^{-4x} \)

Solution:
Form A
\[
\lim_{x \to \infty} 3x^2 e^{-4x} = \lim_{x \to \infty} 3x^2 e^{-4x} = \lim_{x \to \infty} \frac{6x}{4e^4x} = \lim_{x \to \infty} \frac{6}{16e^4x} = 0.
\]

Form B
\[
\lim_{x \to \infty} 2x^2 e^{-6x} = \lim_{x \to \infty} 2x^2 e^{-6x} = \lim_{x \to \infty} \frac{4x}{5e^5x} = \lim_{x \to \infty} \frac{4}{25e^5x} = 0.
\]

(b) (6 points) \( \lim_{h \to 0} \frac{a^{1+h} - a}{h} \)

Solution:
\[
\lim_{h \to 0} \frac{a^{1+h} - a}{h} = \lim_{h \to 0} \frac{\ln(a) a^{1+h}}{1} = \ln(a)a.
\]

14. (6 points) Write \( \lim_{n \to \infty} \sum_{i=1}^{n} e^{i/n} \left( \frac{1}{n} \right) \) as a definite integral and calculate it.

Solution:
\[
\int_{0}^{1} e^x \, dx = e^x \bigg|_{0}^{1} = e - 1.
\]

15. (6 points) Use the definition of the derivative to find \( f'(x) \) when \( f(x) = \sqrt{x} \).

Solution:
\[
f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x} \sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.
\]

16. (8 points) Show \( \ln(x) + x - 2 = 0 \) has exactly one real root in the interval \( (1, e) \).

Solution: If \( f(x) = \ln(x) + x - 2 \), then \( f(1) = -1 < 0 \) and \( f(e) = e - 1 > 0 \). Thus, there is at least one root by the intermediate value theorem.

If \( f \) had two roots, then Rolle’s theorem would tell us that \( f'(c) = 0 \) somewhere between. However,
\[
f'(x) = \frac{1}{x} + 1
\]
is positive on the entire interval. Hence, there is only one root.
17. Evaluate the integrals.

(a) (5 points) \( \int_0^1 v^2 \cos(v^3) \, dv \)

**Solution:** \( u = v^3, \, du = 3v^2 \, dv. \)

\[
\int_0^1 v^2 \cos(v^3) \, dv = \int_0^1 \cos(u) \frac{du}{3} = \frac{1}{3} (\sin(1)).
\]

(b) (5 points) \( \int \frac{\sec \theta \tan \theta}{1 + \sec \theta} \, d\theta \)

**Solution:**

**Form A** \( u = 1 + \sec \theta, \, du = \sec \theta \tan \theta \, d\theta. \)

\[
\int \frac{\sec \theta \tan \theta}{1 + \sec \theta} \, d\theta = \int \frac{du}{u} = \ln |u| + C
\]

\[
= \ln |1 + \sec \theta| + C.
\]

**Form B**

\[
\ln |3 + \sec \theta| + C.
\]

(c) (5 points) \( \int_0^3 |x^2 - 4| \, dx \)

**Solution:**

\[
\int_0^3 |x^2 - 4| \, dx = \int_0^2 4 - x^2 \, dx + \int_2^3 x^2 - 4 \, dx
\]

\[
= (4x - \frac{x^3}{3})|_0^2 + (\frac{x^3}{2} - 4x)|_2^3 = 8 - \frac{8}{3} + \frac{64}{3} - 16 - (\frac{8}{3} - 8)
\]

\[
= \frac{48}{3}
\]