Math 112 (Calculus I)
Final Exam Form A KEY

Part I: Multiple Choice. Enter your answer on the scantron. Work will not be collected or reviewed.

1. Find $\lim_{x \to 2} \left(\frac{x^2 - 2x}{x - 2} \right)$.

a) 1
b) Does not exist
c) -2
d) ∞
e) -1
f) 2
g) 0
h) $-\infty$
i) None of the above.

Solution: f)

2. If for all x you know that $2x^2 + x - 2 \leq f(x) \leq 4x^4 + 2x^2 + x - 2$, do you have enough information to find $\lim_{x \to 0} f(x)$? If so, what is $\lim_{x \to 0} f(x)$?

a) Yes, -2
b) Yes, 0
c) Yes, -1
d) Yes, 2
e) Yes, 1
f) No, not enough information.
g) Yes, but none of the above numbers.

Solution: a)

3. Find $\lim_{x \to \infty} \frac{5 - 3x^3}{\sqrt{81x^6 - 16}}$.

a) Does not exist
b) $-\infty$
c) -3
d) -1
e) $-\frac{1}{3}$
f) 0
g) $\frac{1}{3}$
h) 1
i) 3

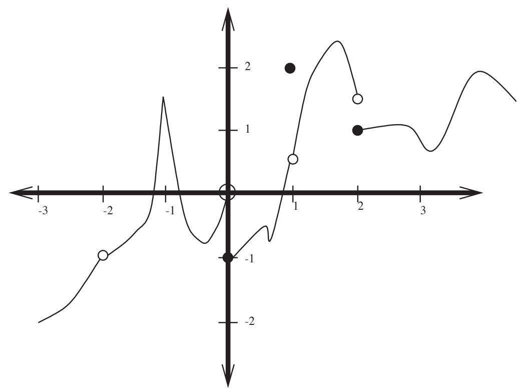
Solution: e)

4. If a function f is defined and twice differentiable on $(-\infty, \infty)$, $f'(2) = 0$, and $f''(2) = 4$, then

a) f has an inflection point at $x = 2$.
b) f is increasing in a neighborhood around $x = 2$.
c) f has a local minimum at $x = 2$.
d) f has a local maximum at $x = 2$.
e) f is decreasing in a neighborhood around $x = 2$.
f) we don't have enough information to prove that any of these are true.

Solution: c)

5. Below is the graph of a function. At which of the following points is it continuous?



a) $x = -1$ b) $x = -2$

c) $x = 2$ d) $x = -1$ and $x = -2$

e) $x = 1$ f) $x = 0$

g) f is not continuous at any of these points. h) f is continuous at all of these points.

Solution: a)

6. Find $f'(x)$ where $f(x) = (x^3 + 5x + 11)^7$.

a) $7(x^3+5x+11)^6(3x^2+5)$ b) $7(x^3 + 5x + 11)^6$ c) $(x^3 + 5x + 11)^7$

d) $(3x^2 + 5)$ e) $7(3x^2 + 5)^6$ f) None of the above.

Solution: a)

7. Let $f(x) = 3x^5 + 5x^4 + 7$. On which of the following intervals is f increasing?

a) $(-4/3, 0)$ b) $(-1, 0)$ c) $(-\infty, -1)$ and $(0, \infty)$

d) $(-1, \infty)$ e) $(-\infty, \infty)$ f) $(-\infty, -4/3)$ and $(0, \infty)$

g) None of these.

Solution: f)

8. What is the maximum y–value of the graph of $f(x) = 4x^2 - x^4 + 1$ on the interval $[-2, 2]$?

a) $y = 2$ b) $y = 9$ c) $y = 5$

d) $y = 6$ e) $y = 0$ f) $y = 4$

g) $y = 1$ h) $y = 3$ i) None of these.

Solution: c)

9. Let $k(x) = \sqrt{x-1}$. For what value of c does $k(x)$ satisfy the Mean Value theorem on the interval $[1, 5]$? (In other words, what value of c satisfies $k'(c) = \dfrac{k(5) - k(1)}{5 - 1}$)?

   a) 1 b) 2 c) 3

   d) 4 e) 5 f) 6

   Solution: b)

10. Let $h(x) = f(g(x))$, and let $g(2) = 1$, $g'(2) = 2$, $f(1) = 3$, $f'(1) = 5$, $f(2) = 3$, and $f'(2) = 7$. Find $h'(2)$.

    a) 14 b) 7 c) 15

    d) 2 e) 21 f) 5

    g) 10 h) 28 i) 35

    j) None of the above.

    Solution: g)

11. Find the derivative $g'(x)$ of the function $g(x) = x^2 \cos x$.

    a) $-2x \sin x$ b) $-\sin 2x$ c) $-2x^3 \sin x \cos x$

    d) $2x \sin x$ e) $2x \cos x - x^2 \sin x$ f) $2x \sin x + x^2 \cos x$

    g) $\cos 2x$ h) None of these.

    Solution: e)

12. Find $\displaystyle\lim_{x \to 2} \left(\frac{|x-2|}{x-2} \right)$.

    a) 1 b) Does not exist c) -1

    d) $-\infty$ e) -2 f) ∞

    g) 0 h) 2

    Solution: b)

13. Find an antiderivative of $f(x) = 3x^2 + \dfrac{2}{x^2}$.

a) $x^3 + \frac{1}{x}$ b) $x^2 + \frac{2}{x^2}$ c) $x^3 - \frac{4}{x^3}$

d) $x^3 + \frac{4}{x^3}$ e) $x^3 - \frac{2}{x}$ f) $x^3 + \frac{2}{x}$

Solution: e)

14. Find $\frac{dy}{dx}$ where $xy = \cos y$.

a) $-\frac{y}{(x + \sin y)}$ b) $-\sin y$ c) $-\frac{\sin y + y}{x}$

d) $\frac{\cos y}{x}$ e) $-\frac{x \sin y + \cos y}{x^2}$ f) None of the above.

Solution: a)

15. Use linear approximation or differentials to estimate $\sqrt[3]{1000.03}$.

a) 10 b) 10.1 c) 10.01

d) 10.001 e) 10.0001 f) None of the above.

Solution: e)

Free response: Write your answer in the space provided. Answers not placed in this space will be ignored.

16. (10 points) **Short answer.** Two points each part. You do not need to show your work on this problem.

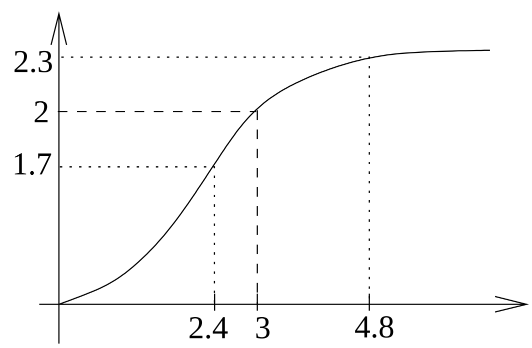
(a) Find $\lim_{x \to 2^+} \frac{1}{x-2}$.

Answer: ______________________

Solution: $+\infty$

(b) Use the given graph of f to find the largest number δ such that

$$\text{if} \quad 0 < |x-3| < \delta \quad \text{then} \quad |f(x) - 2| < 0.3.$$



Answer: ______________________

Solution: $\delta = 0.6$

(c) Find the derivative of $\ln(x-3)$.

Answer: ______________________

Solution: $1/(x-3)$

(d) Integrate $\int \tan x \sec x \, dx$.

Answer: ______________________

Solution: $\int \tan x \sec x \, dx = \sec x + C$

(e) If $\int_1^5 f(x)\,dx = 12$ and $\int_1^4 f(x)\,dx = 4$, what is $\int_4^5 f(x)\,dx$?

Answer: ______________________

Solution: $\int_4^5 f(x)\,dx = 8.$

17. (5 points) Use the definition of the derivative to show:

$$\text{If } f(x) = 3 - 2x^2, \text{ then } f'(x) = -4x.$$

No credit will be given if a method besides the definition of the derivative is used.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{3-2(x+h)^2-(3-2x^2)}{h} \\ &= \lim_{h\to 0} \frac{3-2x^2-4xh-2h^2-3+2x^2}{h} = \lim_{h\to 0} \frac{h(-4x-2h)}{h} \\ &= \lim_{h\to 0} -4x-2h = -4x. \end{aligned}$$

18. (5 points) A snowball is rolling down a hill in such a way that the radius increases steadily by 2 inches every minute. At what rate is its volume increasing when the radius is 10 inches? (Hint: the formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$).

Solution: The formula for the volume of a sphere is:

$$V = \frac{4}{3}\pi r^3$$

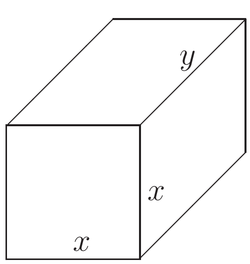
Taking the derivative of each variable with respect to time we find

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

We know that $r = 10$, and $\frac{dr}{dt} = 2$, hence

$$\frac{dV}{dt} = 4 \times \pi \times 10^2 \times 2 = 800\pi \text{ inches}^3 \text{ per second}$$

19. (6 points) Suppose that a post office can accept a package for mailing only if the sum of its length and its girth (the perimeter of its cross section) is at most 120 in. What is the maximum volume of a rectangular box with square cross section that can be mailed?



Solution: 120 = length + girth = L + 2W + 2H.
But since the cross section is a square width = height, which means:

$$120 = L + 4W \text{ which implies } L = 120 - 4W$$

We wish to maximize the volume ($V = L \times W \times H$), so first substitute in for L and H to find volume in terms of width, hence

$$V = W^2(120 - 4W) = 120W^2 - 4W^3$$

Taking the derivative with respect to W, and setting the LHS equal to zero we find

$$0 = 240W - 12W^2 = 12W(20 - W).$$

This implies our critical values are at $W = 0$ and $W = 20$. Taking the second derivative with respect to width we find

$$V''(W) = 240 - 24W$$

When $W = 0$ the second derivative is positive, hence this is a minimum point. When $W = 20$ the second derivative is negative, hence this is a maximum. So the maximum volume occurs when the width is 20 inches. Therefore the maximum volume is $20^2(120-4\times 20) = 400\times 40 = 16000\text{in}^3$.

20. (6 points) Find the derivatives.

(a) Find $f'(x)$ if $f(x) = 7^{x^2}$.

Solution: $f(x) = 7^{x^2} = e^{(\ln 7)x^2}$ using the chain rule $f'(x) = (\ln 7)\times 2x\times e^{(\ln 7)x^2} = 2x(\ln 7)7^{x^2}$.

(b) Find $g'(x)$ if $g(x) = \displaystyle\int_x^{\ln(x)} \frac{1}{2+t^3}dt$.

Solution: Using the fundamental theorem of calculus and the chain rule

$g'(x) = \dfrac{1}{x}\cdot\dfrac{1}{2+(\ln x)^3} - \dfrac{1}{2+x^3}$

21. (6 points) Find the limits.

(a) $\displaystyle\lim_{x\to 0}\frac{\tan x}{\sqrt{x}}$

Solution: Option 1 (L'hospital's rule)

Notice that $\displaystyle\lim_{x\to 0}\frac{\tan x}{\sqrt{x}} = \frac{0}{0}$ Applying L'hospital Rule we find $\displaystyle\lim_{x\to 0}\frac{\tan x}{\sqrt{x}} = \lim_{x\to 0}\frac{\sec^2 x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to 0}\frac{2\sqrt{x}}{\cos^2 x} = 0$

Alternatively,

$$\lim_{x\to 0}\frac{\tan x}{\sqrt{x}} = \lim_{x\to 0}\frac{\sin(x)}{\cos(x)\sqrt{x}} = \lim_{x\to 0}\frac{\sin(x)}{x}\frac{\sqrt{x}}{\cos(x)} = \lim_{x\to 0} 1\cdot\frac{\sqrt{x}}{\cos x} = 0$$

(b) $\displaystyle\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{2i}{n}+6\right)$

Solution: Solution 1: Recognize Riemann sum.

$\displaystyle\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{2i}{n}+6\right) = \int_0^1 (2x+6)dx = x^2 + 6x\Big|_0^1 = 7.$

Solution 2: Use summation formulas.

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{2i}{n}+6\right) &= \lim_{n\to\infty}\frac{1}{n}\left(\frac{2}{n}\sum_{i=1}^{n} i + \sum_{i=1}^{n} 6\right)\\ &= \lim_{n\to\infty}\frac{1}{n}\left(\frac{2}{n}\frac{n(n+1)}{2} + 6n\right)\\ &= \lim_{n\to\infty}\frac{n^2+n}{n^2} + 6 = 7.\end{aligned}$$

22. (6 points) Integrate.

(a) $\int_{-2}^{2} (3x+1)^2\, dx$

Solution: $\int_{-2}^{2} (3x+1)^2\, dx = \int_{-2}^{2} 9x^2 + 6x + 1\, dx = 3x^3 + 3x^2 + x\big]_{-2}^{2} = (3\cdot 8 + 3\cdot 4 + 2) - (3\cdot -8 + 3\cdot 4 - 2) = 52$

(b) $\int \frac{(3-2x)}{(x^2-3x)^{1/3}}\, dx$

Solution: Let $u = x^2 - 3x$ then $du = (2x-3)\, dx$, using u-substitution:

$$\int \frac{(3-2x)}{(x^2-3x)^{1/3}}\, dx = \int -u^{-1/3} du = -\frac{3}{2}u^{2/3} + C = -\frac{3}{2}(x^2-3x)^{2/3} + C$$

23. (5 points) Consider the integral $\int_0^3 (2x+2)\, dx$. Write a Riemann sum approximating the above integral by dividing the interval of integration into n equal parts, and evaluating the function at the right endpoints of the subintervals.

Solution: $a = 0, b = 3$
$\Delta x = \frac{b-a}{n} = \frac{3}{n}$
$f(x_i^*) = 2x_i^* + 2$
$x_i^* = a + i\cdot\Delta x = 0 + i\cdot\frac{3}{n} = \frac{3i}{n}$
Hence the Riemann sum is

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left(\frac{6i}{n} + 2\right)$$

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24. (6 points) In this problem, you will analyze the curve given by

$$f(x) = x^4 - 8x^3 + 18x^2 - 8x + 5.$$

(a) Find all intervals where $f(x)$ is concave up and all intervals where $f(x)$ is concave down.

Solution: $f'(x) = 4x^3 - 24x^2 + 36x - 8$
$f''(x) = 12x^2 - 48x + 36 = 12(x^2 - 4x + 3) = 12(x-1)(x-3)$.
Hence the 2nd derivative is equal to zero when $x = 1$ and when $x = 3$. Now check: when $x < 1$, the second derivative $f'' > 0$; when $1 < x < 3$, the second derivative $f'' < 0$; and when $x > 3$, the second derivative $f'' > 0$.

Hence $f(x)$ is concave up on $(-\infty, 1) \cup (3, \infty)$, and concave down on $(1, 3)$.

(b) At which values of x does $f(x)$ have an inflection point?

Solution: The function has an inflection point at x if f is continuous at x and f changes concavity there. For this function, there are inflection points at 1 and 3.

END OF EXAM