Math 113 – Fall 2005 — key

Departmental Final Exam

Part I: Short Answer and Multiple Choice Questions

Do not show your work for problems in this part.

1. Fill in the blanks with the correct answer.

(a) The integral $\int \cos(x+2)\,dx$ equals $\underline{\int \sin(x+2) + C}$

(b) The integral $\int \sec x \tan x\,dx$ equals $\underline{\sec(x) + C}$

(c) The integral $\int_0^1 \frac{dx}{1+x^2}$ equals $\underline{\tan^{-1} x\Big|_0^1 = \frac{\pi}{4}}$

(d) The integral $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ equals $\underline{\sin^{-1} x\Big|_0^1 = \frac{\pi}{2}}$

(e) The integral $\int \tan^2 x\,dx$ equals $\underline{\int \sec^2(x) - 1\,dx = \tan(x) - x + C}$

(f) The integral $\int_0^1 \frac{dx}{\sqrt{x}}$ equals $\underline{2\sqrt{x}\Big|_0^1 = 2}$

(g) The integral $\int_0^\infty \frac{dx}{x^3}$ equals <u>divergent integral</u>

(h) The integral $\int \frac{x}{\sqrt{1+x^2}}\,dx$ equals $\underline{\sqrt{1+x^2} + C}$

(i) Give the limit of the sequence $\left\{\left(1-\frac{1}{n}\right)^n\right\}$ as $n \to \infty$ if it is convergent, otherwise write DIVERGENT.

$\underline{e^{-1}}$

(j) State the integration by parts formula:

$\underline{\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx}$

(k) Give a limit definition of the improper integral $\int_0^1 \frac{\sin x}{\sqrt{x}}\,dx$

$\underline{\lim_{\epsilon\to 0} \int_\epsilon^1 \frac{\sin x}{\sqrt{x}}\,dx}$

(l) State the $(2m)$-th term of the MacLaurin series for $\frac{\sin x}{x}$

$\underline{\frac{(-1)^m}{(2m+1)!}x^{2m}}$

(m) The integral $\int \cot x\,dx$ equals $\underline{\ln(\sin(x)) + C}$

2. True/False: Write T if statement always holds, F otherwise.

Let $\sum a_n = \sum_{n=1}^{\infty} a_n$ be an arbitrary series.

(a) F : need $a_n \to 0$ If $\{a_n\}$ is a positive decreasing sequence then $\sum(-1)^n a_n$ converges.

(b) T: Divergence test If $\sum a_n$ converges then $a_n \to 0$.

(c) F : $1 - 1 + 1 - 1...$ If the partial sums of $\sum a_n$ are bounded, then $\sum a_n$ converges.

Problems 3 through 9 are multiple choice. Each multiple choice problem is worth 3 points. In the grid below fill in the square corresponding to each correct answer.

3. The most appropriate first step to integrate $\int \frac{x^2 - 1}{3x^3 - x^2}\, dx$ would be

(a) Integration by parts

(b) Partial fractions

(c) Trigonometric Substitution

(d) Other (non trigonometric) substitution

(e) Differentiate the integrand

(f) None of these

4. The series $x^2 + x^4 + \frac{x^6}{2} + \frac{x^8}{6} + \cdots = \sum_{n=0}^{\infty} \frac{x^{(2n+2)}}{n!}$ converges to the function

(a) $\frac{x^2}{1+x^2}$

(b) $x^2 \tan^{-1} x$

(c) e^{x^2+2}

(d) $x^2 e^{x^2}$

(e) $x^2(\sin x^2 + \cos x^2)$

(f) $\sin x^2 + \cos x^2$

(g) None of these

5. The improper integral $\int_0^{\infty} xe^{-x} dx$ converges to

(a) 0

(b) $1/e$

(c) $1/2$

(d) 1

(e) 2

(f) e

(g) None of these

(h) It doesn't converge

6. The length of the curve $y = \cosh x$ from $x = 0$ to $x = 1$ is

(a) $\boxed{\sinh 1}$

(b) $\cosh 1$

(c) $\cosh^2 1 - \cosh^2 0$

(d) 1

(e) ∞

(f) a real number in (0,1)

(g) Imaginary

(h) None of these

7. The area enclosed by the polar curve $r = 3 + \sin\theta$ is

(a) 5π

(b) 4π

(c) 9π

(d) $\pi/4$

(e) 4.5π

(f) 19π

(g) $9\pi^2$

(h) $\boxed{\text{None of these: } 19\pi/2}$

8. The interval of convergence of the power series $\sum_{n=1}^{\infty} n^2(5x-3)^n$ is

(a) $(-3/5, 3/5)$

(b) $(-5/3, 5/3)$

(c) $(0, 1)$

(d) $(-1, 1)$

(e) $\boxed{(2/5, 4/5)}$

(f) $(1/5, 1)$

(g) $(0, \infty)$

(h) $(-\infty, \infty)$

(i) None of the above

9. The coefficient of x^3 in the series expansion of $(1+x)^{1/4}$ is

(a) $\frac{1}{4^3} = \frac{1}{64}$

(b) $\frac{1}{4^3 3!} = \frac{1}{384}$

(c) $\frac{6}{4^3 3!} = \frac{1}{64}$

(d) $\frac{15}{4^3 3!} = \frac{5}{128}$

(e) $\frac{20}{4^3 3!} = \frac{5}{96}$

(f) $\boxed{\frac{21}{4^3 3!} = \frac{7}{128}}$

(g) $\frac{25}{4^3 3!} = \frac{25}{384}$

(h) $\frac{35}{4^3 3!} = \frac{35}{384}$

(i) None of the above

The answers to the multiple choice MUST be entered on the grid on the previous page. Otherwise, you will not receive credit.

Part II: Written Solutions

For problems 10 – 18, write your answers in the space provided. Neatly show your work for full credit.

10. (a) Evaluate the integral $\int_0^1 t^2 e^t\,dt$.

Let $u = t^2,\ dv = e^t\,dt$, then $du = 2t\,dt,\ v = e^t$,

$$\int_0^1 t^2 e^t\,dt = t^2 e^t\Big|_0^1 - \int_0^1 2te^t\,dt$$

Let $u = 2t,\ dv = e^t\,dt$, then $du = 2dt,\ v = e^t$,

$$\begin{aligned}\int_0^1 t^2 e^t\,dt &= e - \left(2te^t\Big|_0^1 - \int_0^1 2e^t\,dt\right)\\ &= e - (2e - 2(e-1))\\ &= e - 2\end{aligned}$$

(b) Expand in partial fraction form $\frac{x^2+3}{x^2-1}$.

$$\frac{x^2+3}{x^2-1} = \frac{x^2-1+4}{x^2-1} = 1 + \frac{4}{(x-1)(x+1)} = 1 + 2\,\frac{1}{x-1} - 2\,\frac{1}{1+x}$$

(c) Evaluate the integral $\int \frac{x^2+3}{x^2-1}\,dx$.

$$x + 2\ln\left(\frac{x-1}{x+1}\right)$$

11. Evaluate the integral $\int \frac{1}{4-3\sin x}\, dx$. Let $z = \tan(x/2)$, then

$$dx = \frac{2\,dz}{1+z^2}, \quad \sin x = \frac{2z}{1+z^2}$$

$$\begin{aligned}
\int \frac{1}{4-3\sin x}\, dx &= \int \frac{1}{4-3\frac{2z}{1+z^2}} \frac{2\,dz}{1+z^2} \\
&= \int \frac{2}{4z^2-6z+4}\, dz = \int \frac{1}{2z^2-3z+2}\, dz \\
&= \frac{1}{2}\int \frac{1}{\left(z-\frac{3}{4}\right)^2 + \left(\frac{\sqrt{7}}{4}\right)^2}\, dz
\end{aligned}$$

Let $z - \frac{3}{4} = \frac{\sqrt{7}}{4}\tan t$, then $dz = \frac{\sqrt{7}}{4}\sec^2 t\, dt$ and

$$\begin{aligned}
\int \frac{1}{4-3\sin x}\, dx &= \int \frac{2}{\sqrt{7}}\, dt \\
&= \frac{2}{\sqrt{7}} t + C \\
&= \frac{2}{\sqrt{7}} \tan^{-1}\left(\frac{4}{\sqrt{7}}\left(z - \frac{3}{4}\right)\right) + C \\
&= \frac{2}{\sqrt{7}} \tan^{-1}\left(\frac{4}{\sqrt{7}}\left(\tan\left(\frac{x}{2}\right) - \frac{3}{4}\right)\right) + C \\
&= \frac{2}{\sqrt{7}} \tan^{-1}\left(\frac{1}{\sqrt{7}}\left(4\tan\left(\frac{x}{2}\right) - 3\right)\right) + C
\end{aligned}$$

12. The region bounded by $y = x$ and $y = 2x^2$ is revolved about the **y-axis** ; find the volume of the solid generated.

Intersection of curves:

$$x = 2x^2 \Rightarrow x = 0, \frac{1}{2} \Rightarrow \text{points of intersection: } (0,0), \left(\frac{1}{2}, \frac{1}{2}\right)$$

Disc method:

$$\int_0^{1/2} \pi \left(\sqrt{\frac{y}{2}}\right)^2 - \pi y^2 \, dy = \pi \int_0^{1/2} \frac{y}{2} - y^2 \, dy = \pi \left[\frac{1}{4} y^2 - \frac{1}{3} y^3\right]_0^{1/2} = \frac{\pi}{48}$$

Shell method:

$$\int_0^{1/2} 2\pi x(x - 2x^2) \, dx = \pi \left[-x^4 + \frac{2}{3} x^3\right]_0^{1/2} = \frac{\pi}{48}$$

13. Find the area of the surface of revolution generated by revolving the curve $y = \sqrt{x}$, $0 \le x \le 4$, about the x-axis.

$$\begin{aligned}
\text{Area} &= \int_0^4 2\pi\sqrt{x}\sqrt{1 + \left(\frac{d\sqrt{x}}{dx}\right)^2} \, dx \\
&= \int_0^4 2\pi\sqrt{x}\sqrt{1 + \frac{1}{4x}} \, dx \\
&= \pi \int_0^4 \sqrt{1 + 4x} \, dx \\
&= \frac{\pi}{4} \cdot \frac{2}{3}(1 + 4x)^{3/2}\Big|_0^4 \\
&= \frac{\pi}{6}\left(17^{3/2} - 1\right)
\end{aligned}$$

14. Find the centroid of the region bounded by the curves

$$y = \sqrt{1+x^2}, \;\; x = 1 \;\; \text{and} \;\; y = 1+x.$$

Express you answer in terms of unevaluated integrals. (Note: You should simplify the integrands as much as possible.)

The curves $y = \sqrt{1+x^2}$ and $y = 1+x$ intersects at $x = 0$.

Area of region

$$A = \int_0^1 1 + x - \sqrt{1+x^2}\, dx$$

Coordinates of centroid $(\bar{x}, \bar{y})$:

$$\bar{x} = \frac{\int_0^1 x(1 + x - \sqrt{1+x^2})\, dx}{A}$$

$$\begin{aligned}
\bar{y} &= \frac{\int_0^1 \frac{1}{2}\left(1 + x + \sqrt{1+x^2}\right)\left(1 + x - \sqrt{1+x^2}\right)\, dx}{A} \\
&= \frac{\int_0^1 \frac{1}{2}\left((1+x)^2 - (1+x^2)\right)\, dx}{A} \\
&= \frac{1}{A}\int_0^1 x\, dx \quad \left(= \frac{1}{2}A\right)
\end{aligned}$$

15. If a region in the first quadrant, with area 10π and centroid at the point $(1, 12)$, is revolved around the line $x = -5$, find the resulting volume of revolution.

By the first theorem of Pappus, $V = 2\pi\bar{r}A$.

Now $A = 10\pi$, $\bar{r} = 1 - (-5) = 6$, so

$$V = 120\pi^2$$

16. Determine whether each infinite series is absolutely convergent, conditionally convergent, or divergent. Give reasons for your conclusion.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{3n+7}$

Divergent: comparison with harmonic series

$$\sum_{n=3}^{\infty} \frac{\ln n}{3n+7} \geq \sum_{n=3}^{\infty} \frac{1}{3n+7} \geq \sum_{n=3}^{\infty} \frac{1}{6n} = \frac{1}{6}\sum_{n=3}^{\infty} \frac{1}{n}$$

(b) $\displaystyle\sum_{n=1}^{\infty} (3^{-n} - 5^{-n})$

Absolutely convergent: geometric series

$$\sum_{n=1}^{\infty} (3^{-n} - 5^{-n}) = \sum_{n=1}^{\infty} 3^{-n} - \sum_{n=1}^{\infty} 5^{-n}) = \frac{1/3}{1-1/3} - \frac{1/5}{1-1/5} == \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

Conditionally convergent: alternating series related to decreasing sequence of positive terms and comparison test

Series is convergent since the sequence $\left\{\dfrac{1}{n \ln n}\right\}$ is a decreasing sequence of positive terms, hence the alternating series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ is convergent.

Series is not absolutely convergent since the integral $\displaystyle\int_2^{\infty} \frac{1}{x \ln x}\, dx$ is divergent.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{\ln(2n)}$

Divergent: divergence test

$$\lim_{n\to\infty} \frac{(-1)^n n}{\ln(2n)} = \lim_{n\to\infty} \frac{(-1)^n}{2/n} = \pm\infty \neq 0$$

17. (a) Determine the power series expansion of $\int \tan^{-1} x \, dx$.

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 + \cdots + \frac{(-1)^n}{2n+1}x^{2n+1} + \cdots$$

so

$$\int \tan^{-1} x \, dx = C + \frac{x^2}{2} - \frac{1}{12}x^4 + \frac{1}{30}x^6 \cdots + \frac{(-1)^n}{(2n+1)(2n+2)}x^{2n+2} + \cdots$$

(b) Find first two nonzero terms of the Taylor series of $\ln(1 + \sin^2 x)$ at $x = \pi$. What is the remainder after these terms?

From Maclaurin expansion,

$$\ln(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \cdots + \frac{(-1)^{n+1}}{n}z^n + \cdots$$

As $\sin^2 \pi = 0$, so

$$\ln(1 + \sin^2 x) = \sin^2 x - \frac{1}{2}\sin^4 x + \frac{1}{3}\sin^6 x + \cdots$$

Also the Taylor series of $\sin x$ about $x = \pi$ is

$$\sin x = -(x-\pi) + \frac{1}{6}(x-\pi)^3 - \frac{1}{120}(x-\pi)^5 + \cdots$$

Thus

$$\sin^2 x = (x-\pi)^2 - \frac{1}{3}(x-\pi)^4 + \cdots$$

Hence

$$\begin{aligned} \ln(1+\sin^2 x) &= (x-\pi)^2 - \frac{1}{3}(x-\pi)^4 - \frac{1}{2}\left((x-\pi)^2 - \frac{1}{3}(x-\pi)^4 + \cdots\right)^2 + \cdots \\ &= (x-\pi)^2 - \frac{5}{6}(x-\pi)^4 + \cdots \end{aligned}$$

The remainder is given by

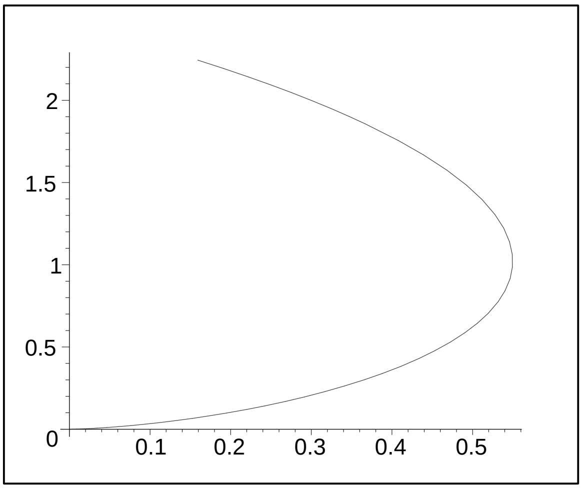
$$\int_\pi^x \frac{1}{5!}\frac{d^6}{dx^6}\ln(1+\sin^2 t)\,(x-t)^5\,dt \quad \text{or} \quad \frac{1}{6!}\left.\frac{d^6}{dx^6}\ln(1+\sin^2 x)\right|_{x=\xi}(x-\pi)^6$$

where ξ is a number between x and π.

18. Given the polar curve $r = \theta^2,\ 0 \le \theta \le 3/2$,

(a) sketch the curve;

$\frac{3}{2}$ radian is slightly less than $\frac{\pi}{2}$ radian or $90°$. (note: $\frac{3}{2}$ radian is about $86°$)



(b) find the area swept out by the curve;

$$\text{Area} = \int_0^{3/2} \frac{1}{2} r^2 \, d\theta = \int_0^{3/2} \frac{1}{2} \theta^4 \, d\theta = \frac{1}{10}\left(\frac{3}{2}\right)^5$$

(c) find the arc length.

$$\begin{aligned} \text{Arc length} &= \int_0^{3/2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta \\ &= \int_0^{3/2} \sqrt{\theta^4 + 4\theta^2} \, d\theta \\ &= \int_0^{3/2} \theta\sqrt{\theta^2 + 4} \, d\theta \\ &= \frac{1}{3} \left(t^2 + 4\right)^{3/2} \Big|_0^{3/2} \\ &= \frac{125}{24} - \frac{8}{3} \\ &= \frac{61}{24} \end{aligned}$$

—*End*—