Math 113 – Fall 2005 — key
Departmental Final Exam

PART I: SHORT ANSWER AND MULTIPLE CHOICE QUESTIONS
Do not show your work for problems in this part.

1. Fill in the blanks with the correct answer.

(a) The integral \( \int \cos(x + 2) \, dx \) equals \( \sin(x + 2) + C \)
(b) The integral \( \int \sec x \tan x \, dx \) equals \( \sec(x) + C \)
(c) The integral \( \int_0^1 \frac{dx}{1 + x^3} \) equals \( \tan^{-1} x \bigg|_0^1 = \frac{\pi}{4} \)
(d) The integral \( \int_0^1 \frac{dx}{\sqrt{1 - x^2}} \) equals \( \sin^{-1} x \bigg|_0^1 = \frac{\pi}{2} \)
(e) The integral \( \int \tan^2 x \, dx \) equals \( \int \sec^2(x) - 1 \, dx = \tan(x) - x + C \)
(f) The integral \( \int_0^1 \frac{dx}{\sqrt{x}} \) equals \( 2\sqrt{x} \bigg|_0^1 = 2 \)
(g) The integral \( \int_0^\infty \frac{dx}{x^3} \) equals divergent integral
(h) The integral \( \int \frac{x}{\sqrt{1 + x^2}} \, dx \) equals \( \sqrt{1 + x^2} + C \)
(i) Give the limit of the sequence \( \left\{ \left(1 - \frac{1}{n}\right)^n \right\} \) as \( n \to \infty \) if it is convergent, otherwise write DIVERGENT.
\( e^{-1} \)
(j) State the integration by parts formula:
\[ \int u(x)v'(x) \, dx = u(x)v(x) - \int u'(x)v(x) \, dx \]
(k) Give a limit definition of the improper integral \( \int_0^1 \frac{\sin x}{\sqrt{x}} \, dx \)
\( \lim_{\epsilon \to 0} \int_{\epsilon}^1 \frac{\sin x}{\sqrt{x}} \, dx \)
(l) State the \((2m)\)-th term of the MacLaurin series for \( \frac{\sin x}{x} \)
\[ \frac{(-1)^m}{(2m + 1)!} x^{2m} \]
(m) The integral \( \int \cot x \, dx \) equals \( \ln(\sin(x)) + C \)
2. True/False: Write T if statement always holds, F otherwise.

Let \( \sum a_n = \sum_{n=1}^{\infty} a_n \) be an arbitrary series.

(a) \(F:\) need \( a_n \to 0 \) If \( \{a_n\} \) is a positive decreasing sequence then \( \sum (-1)^n a_n \) converges.
(b) \(T:\) Divergence test If \( \sum a_n \) converges then \( a_n \to 0 \).
(c) \(F:\) \( 1 - 1 + 1 - 1... \) If the partial sums of \( \sum a_n \) are bounded, then \( \sum a_n \) converges.

Problems 3 through 9 are multiple choice. Each multiple choice problem is worth 3 points. In the grid below fill in the square corresponding to each correct answer.

3. The most appropriate first step to integrate \( \int \frac{x^2 - 1}{3x^3 - x^2} \, dx \) would be

(a) Integration by parts (d) Other (non trigonometric) substitution
(b) Partial fractions (e) Differentiate the integrand
(c) Trigonometric Substitution (f) None of these

4. The series \( x^2 + x^4 + \frac{x^6}{6} + \frac{x^8}{8} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{n!} \) converges to the function

(a) \( \frac{x^2}{1+x^2} \) (e) \( x^2(\sin x^2 + \cos x^2) \)
(b) \( x^2 \tan^{-1} x \) (f) \( \sin x^2 + \cos x^2 \)
(c) \( e^{x^2+2} \) (g) None of these
(d) \[x^2e^{x^2}\]

5. The improper integral \( \int_0^{\infty} xe^{-x} \, dx \) converges to

(a) 0 (e) 2
(b) \( 1/e \) (f) \( e \)
(c) \( 1/2 \) (g) None of these
(d) \( 1 \) (h) It doesn’t converge
6. The length of the curve \( y = \cosh x \) from \( x = 0 \) to \( x = 1 \) is
   (a) \( \sinh 1 \)  
   (b) \( \cosh 1 \)  
   (c) \( \cosh^2 1 - \cosh^2 0 \)  
   (d) \( 1 \)  
   (e) \( \infty \)  
   (f) a real number in (0,1)  
   (g) Imaginary  
   (h) None of these

7. The area enclosed by the polar curve \( r = 3 + \sin \theta \) is
   (a) \( 5\pi \)  
   (b) \( 4\pi \)  
   (c) \( 9\pi \)  
   (d) \( \pi/4 \)  
   (e) \( 4.5\pi \)  
   (f) \( 19\pi \)  
   (g) \( 9\pi^2 \)  
   (h) None of these: \( 19\pi/2 \)

8. The interval of convergence of the power series \( \sum_{n=1}^{\infty} n^2 (5x - 3)^n \) is
   (a) \((-3/5, 3/5)\)  
   (b) \((-5/3, 5/3)\)  
   (c) \((0, 1)\)  
   (d) \((-1, 1)\)  
   (e) \((2/5, 4/5)\)  
   (f) \((1/5, 1)\)  
   (g) \((0, \infty)\)  
   (h) \((-\infty, \infty)\)  
   (i) None of the above

9. The coefficient of \( x^3 \) in the series expansion of \( (1 + x)^{1/4} \) is
   (a) \( \frac{1}{4^3} = \frac{1}{64} \)  
   (b) \( \frac{1}{4^33!} = \frac{1}{384} \)  
   (c) \( \frac{6}{4^33!} = \frac{1}{64} \)  
   (d) \( \frac{15}{4^33!} = \frac{5}{128} \)  
   (e) \( \frac{20}{4^33!} = \frac{5}{96} \)  
   (f) \( \frac{21}{4^33!} = \frac{7}{128} \)  
   (g) \( \frac{25}{4^33!} = \frac{25}{584} \)  
   (h) \( \frac{35}{4^33!} = \frac{35}{584} \)  
   (i) None of the above

The answers to the multiple choice MUST be entered on the grid on the previous page. Otherwise, you will not receive credit.
10. (a) Evaluate the integral \( \int_0^1 t^2 e^t \, dt \).

Let \( u = t^2 \), \( dv = e^t \, dt \), then \( du = 2t \, dt \), \( v = e^t \),

\[
\int_0^1 t^2 e^t \, dt = t^2 e^t \bigg|_0^1 - \int_0^1 2te^t \, dt
\]

Let \( u = 2t \), \( dv = e^t \, dt \), then \( du = 2dt \), \( v = e^t \),

\[
\int_0^1 t^2 e^t \, dt = e - \left( 2te^t \bigg|_0^1 - \int_0^1 2e^t \, dt \right)
\]
\[
= e - 2(e - 2(e - 1))
\]
\[
= e - 2
\]

(b) Expand in partial fraction form \( \frac{x^2 + 3}{x^2 - 1} \).

\[
\frac{x^2 + 3}{x^2 - 1} = \frac{x^2 - 1 + 4}{x^2 - 1} = 1 + \frac{4}{(x-1)(x+1)} = 1 + 2 \frac{1}{x-1} - 2 \frac{1}{1+x}
\]

(c) Evaluate the integral \( \int \frac{x^2 + 3}{x^2 - 1} \, dx \).

\[
x + 2 \ln \left( \frac{x - 1}{x + 1} \right)
\]
11. Evaluate the integral \( \int \frac{1}{4 - 3 \sin x} \, dx \). Let \( z = \tan(x/2) \), then

\[ dx = \frac{2 \, dz}{1 + z^2}, \quad \sin x = \frac{2z}{1 + z^2} \]

\[ \int \frac{1}{4 - 3 \sin x} \, dx = \int \frac{1}{4 - 3 \frac{2z}{1 + z^2}} \frac{2 \, dz}{1 + z^2} \]

\[ = \int \frac{2}{4z^2 - 6z + 4} \, dz = \int \frac{1}{2z^2 - 3z + 2} \, dz \]

\[ = \frac{1}{2} \int \frac{1}{(z - \frac{3}{4})^2 + \left(\frac{\sqrt{7}}{4}\right)^2} \, dz \]

Let \( z - \frac{3}{4} = \frac{\sqrt{7}}{4} \tan t \), then \( dz = \frac{\sqrt{7}}{4} \sec^2 t \, dt \) and

\[ \int \frac{1}{4 - 3 \sin x} \, dx = \int \frac{2}{\sqrt{7}} \, dt \]

\[ = \frac{2}{\sqrt{7}} t + C \]

\[ = \frac{2}{\sqrt{7}} \tan^{-1} \left( \frac{4}{\sqrt{7}} \left( z - \frac{3}{4} \right) \right) + C \]

\[ = \frac{2}{\sqrt{7}} \tan^{-1} \left( \frac{4}{\sqrt{7}} \left( \tan \left( \frac{x}{2} \right) - \frac{3}{4} \right) \right) + C \]

\[ = \frac{2}{\sqrt{7}} \tan^{-1} \left( \frac{1}{\sqrt{7}} \left( 4 \tan \left( \frac{x}{2} \right) - 3 \right) \right) + C \]
12. The region bounded by $y = x$ and $y = 2x^2$ is revolved about the $y$-axis; find the volume of the solid generated.

Intersection of curves:

$$x = 2x^2 \Rightarrow x = 0, \frac{1}{2} \Rightarrow \text{points of intersection: } (0,0), \left(\frac{1}{2}, \frac{1}{2}\right)$$

Disc method:

$$\int_0^{1/2} \pi \left(\sqrt{\frac{y}{2}}\right)^2 - \pi y^2 dy = \pi \int_0^{1/2} \frac{y}{2} - y^2 dy = \pi \left[\frac{1}{4} y^2 - \frac{1}{3} y^3\right]_0^{1/2} = \frac{\pi}{48}$$

Shell method:

$$\int_0^{1/2} 2\pi x (x - 2x^2) dx = \pi \left[-x^4 + \frac{2}{3} x^3\right]_0^{1/2} = \frac{\pi}{48}$$

13. Find the area of the surface of revolution generated by revolving the curve $y = \sqrt{x}, 0 \leq x \leq 4$, about the $x$-axis.

Area $= \int_0^4 2\pi \sqrt{x} \sqrt{1 + \left(\frac{d\sqrt{x}}{dx}\right)^2} dx$

$= \int_0^4 2\pi \sqrt{x} \sqrt{1 + \frac{1}{4x}} dx$

$= \pi \int_0^4 \sqrt{1 + \frac{4}{4x}} dx$

$= \pi \left[\frac{2}{3} (1 + 4x)^{3/2}\right]_0^4$

$= \frac{\pi}{6} (17^{3/2} - 1)$
14. Find the centroid of the region bounded by the curves

\[ y = \sqrt{1 + x^2}, \quad x = 1 \quad \text{and} \quad y = 1 + x. \]

Express your answer in terms of unevaluated integrals. (Note: You should simplify the integrands as much as possible.)

The curves \( y = \sqrt{1 + x^2} \) and \( y = 1 + x \) intersect at \( x = 0 \).

Area of region

\[ A = \int_0^1 1 + x - \sqrt{1 + x^2} \, dx \]

Coordinates of centroid \( (\bar{x}, \bar{y}) \):

\[
\bar{x} = \frac{\int_0^1 x(1 + x - \sqrt{1 + x^2}) \, dx}{A}
\]

\[
\bar{y} = \frac{\int_0^1 \frac{1}{2} \left(1 + x + \sqrt{1 + x^2}\right) \left(1 + x - \sqrt{1 + x^2}\right) \, dx}{A}
= \frac{\int_0^1 \frac{1}{2} ((1 + x)^2 - (1 + x^2)) \, dx}{A}
= \frac{1}{A} \int_0^1 x \, dx \quad (= \frac{1}{2} \bar{A})
\]

15. If a region in the first quadrant, with area \( 10\pi \) and centroid at the point \( (1, \frac{12}{2}) \), is revolved around the line \( x = -5 \), find the resulting volume of revolution.

By the first theorem of Pappus, \( V = 2\pi \bar{r} A \).

Now \( A = 10\pi \), \( \bar{r} = 1 - (-5) = 6 \), so

\[ V = 120\pi^2 \]
16. Determine whether each infinite series is absolutely convergent, conditionally convergent, or divergent. Give reasons for your conclusion.

(a) $\sum_{n=1}^{\infty} \frac{\ln n}{3n + 7}$

Divergent: comparison with harmonic series

$$\sum_{n=3}^{\infty} \frac{\ln n}{3n + 7} \geq \sum_{n=3}^{\infty} \frac{1}{3n + 7} \geq \sum_{n=3}^{\infty} \frac{1}{6n} = \frac{1}{6} \sum_{n=3}^{\infty} \frac{1}{n}$$

(b) $\sum_{n=1}^{\infty} (3^{-n} - 5^{-n})$

Absolutely convergent: geometric series

$$\sum_{n=1}^{\infty} (3^{-n} - 5^{-n}) = \sum_{n=1}^{\infty} 3^{-n} - \sum_{n=1}^{\infty} 5^{-n} = \frac{1/3}{1 - 1/3} - \frac{1/5}{1 - 1/5} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

(c) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

Conditionally convergent: alternating series related to decreasing sequence of positive terms and comparison test

Series is convergent since the sequence $\left\{ \frac{1}{n \ln n} \right\}$ is a decreasing sequence of positive terms, hence the alternating series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ is convergent.

Series is not absolutely convergent since the integral $\int_{2}^{\infty} \frac{1}{x \ln x} \, dx$ is divergent.

(d) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{\ln(2n)}$

Divergent: divergence test

$$\lim_{n \to \infty} \frac{(-1)^n n}{\ln(2n)} = \lim_{n \to \infty} \frac{(-1)^n n}{2/n} = \pm \infty \neq 0$$
17. (a) Determine the power series expansion of \( \int \tan^{-1} x \, dx \).

\[
\tan^{-1} x = x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \frac{1}{7} x^7 + \frac{1}{9} x^9 + \cdots + \frac{(-1)^n}{2n+1} x^{2n+1} + \cdots
\]

so

\[
\int \tan^{-1} x \, dx = C + \frac{x^2}{2} - \frac{1}{12} x^4 + \frac{1}{30} x^6 \cdots + \frac{(-1)^n}{(2n+1)(2n+2)} x^{2n+2} + \cdots
\]

(b) Find first two nonzero terms of the Taylor series of \( \ln(1 + \sin^2 x) \) at \( x = \pi \). What is the remainder after these terms?

From Maclaurin expansion,

\[
\ln(1 + z) = z - \frac{1}{2} z^2 + \frac{1}{3} z^3 - \frac{1}{4} z^4 + \cdots + \frac{(-1)^{n+1}}{n} z^n + \cdots
\]

As \( \sin^2 \pi = 0 \), so

\[
\ln(1 + \sin^2 x) = \sin^2 x - \frac{1}{2} \sin^4 x + \frac{1}{3} \sin^6 x + \cdots
\]

Also the Taylor series of \( \sin x \) about \( x = \pi \) is

\[
\sin x = -(x - \pi) + \frac{1}{6} (x - \pi)^3 - \frac{1}{120} (x - \pi)^5 + \cdots
\]

Thus

\[
\sin^2 x = (x - \pi)^2 - \frac{1}{3} (x - \pi)^4 + \cdots
\]

Hence

\[
\ln(1 + \sin^2 x) = (x - \pi)^2 - \frac{1}{3} (x - \pi)^4 - \frac{1}{2} \left( (x - \pi)^2 - \frac{1}{3} (x - \pi)^4 + \cdots \right)^2 + \cdots
\]

\[
= (x - \pi)^2 - \frac{5}{6} (x - \pi)^4 + \cdots
\]

The remainder is given by

\[
\int_{\pi}^{x} \frac{1}{5!} \frac{d^6}{dx^6} \ln(1 + \sin^2 t) (x - t)^5 \, dt \quad \text{or} \quad \frac{1}{6!} \frac{d^6}{dx^6} \ln(1 + \sin^2 x) \bigg|_{x=\xi} \quad (x - \pi)^6
\]

where \( \xi \) is a number between \( x \) and \( \pi \).
18. Given the polar curve \( r = \theta^2 \), \( 0 \leq \theta \leq 3/2 \),

(a) sketch the curve;

\( \frac{3}{2} \) radian is slightly less than \( \frac{\pi}{2} \) radian or \( 90^\circ \). (note: \( \frac{3}{2} \) radian is about \( 86^\circ \) )

(b) find the area swept out by the curve;

\[
\text{Area} = \int_0^{3/2} \frac{1}{2} r^2 d\theta = \int_0^{3/2} \frac{1}{2} \theta^4 d\theta = \frac{1}{10} \left( \frac{3}{2} \right)^5
\]

(c) find the arc length.

\[
\text{Arc length} = \int_0^{3/2} \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} d\theta
\]

\[
= \int_0^{3/2} \sqrt{\theta^4 + 4\theta^2} d\theta
\]

\[
= \int_0^{3/2} \theta \sqrt{\theta^2 + 4} d\theta
\]

\[
= \frac{1}{3} \left( t^2 + 4 \right)^{3/2} \bigg|_0^{3/2}
\]

\[
= \frac{125}{24} - \frac{8}{3}
\]

\[
= \frac{61}{24}
\]

--- End ---