Math 113 – Fall 2006 – Key

Departmental Final Exam

Part I: Short Answer and Multiple Choice Questions

Do not show your work for problem 1.

1. Fill in the blanks with the correct answer.

   (a) Does the improper integral $\int_0^\infty \frac{dx}{e^x+1}$ converge (yes or no) <u>*yes*</u>

   (b) The integral $\int \frac{\cos x}{\sin^3 x}\,dx$ equals $\underline{-\frac{1}{2\sin^2 x}+C}$

   (c) The integral $\int_1^{e^2} \frac{dx}{2x}$ equals $\underline{1}$

   (d) $\frac{x^2}{4}-\frac{y^2}{25}=1$ is the equation of a/an <u>*hyperbola*</u>

   (e) The radius of convergence of $\sum_{n=0}^{\infty} 3^n x^n$ is $\underline{\frac{1}{3}}$

   (f) If $n>1$, the integral $\int_1^\infty \frac{dx}{x^n}$ equals $\underline{\frac{1}{n-1}}$

   (g) The series $x^2-\frac{x^4}{3!}+\frac{x^6}{5!}-\frac{x^8}{7!}+\ldots$ is the MacLaurin series for the function $\underline{x\sin x}$

   (h) The integral $\int x\ \sin x\,dx$ equals $\underline{-x\cos x+\sin x+C}$

   (i) The series $2-\frac{2}{3}+\frac{2}{9}-\frac{2}{27}+\ldots$ converges to $\underline{\frac{3}{2}}$

   Problems 2 through 8 are multiple choice. Each multiple choice problem is worth 3 points. In the grid below fill in the square corresponding to each correct answer.

2. Which of the following integrals represents the surface area of the surface generated by revolving the curve $y=\tan x$, $0\le x\le \pi/4$, about the line $y=-2$?

   (a) $\int_0^{\pi/4} \pi(\tan x\ +2)\sqrt{1+\sec^2 x}\,dx$

   (b) $\int_0^{\pi/4} 2\pi(\tan x\ +2)\sqrt{1+\sec^2 x}\,dx$

   (c) $\int_0^{\pi/4} \pi(\tan x\ +2)\sqrt{1+\sec^4 x}\,dx$

   [(d)] $\int_0^{\pi/4} 2\pi(\tan x\ +2)\sqrt{1+\sec^4 x}\,dx$

   (e) $\int_0^{\pi/4} \pi(\tan x\ -2)\sqrt{1+\sec^4 x}\,dx$

   (f) $\int_0^{\pi/4} 2\pi(\tan x\ -2)\sqrt{1+\sec^2 x}\,dx$

   (g) $\int_0^{\pi/4} \pi(\tan x\ -2)\sqrt{1+\sec^2 x}\,dx$

   (h) $\int_0^{\pi/4} 2\pi(\tan x\ -2)\sqrt{1+\sec^4 x}\,dx$

   (i) None of the above

3. Which of the following substitutions will best simplify the integral $\int \sqrt{3+2x-x^2}\,dx$?

(a) $x = 1 - 2\sec u$

(b) $x = \sqrt{3} + 2\cosh u$

(c) $x = \sqrt{3}\cos u$

(d) $x = \sqrt{3} - 2\cosh u$

(e) $x = \sqrt{3}\sin u$

$\boxed{\text{(f)}}$ $x = 1 + 2\sin u$

(g) $x = 2\sin u$

4. Consider the region R that is the portion of the circle $x^2+y^2 = 1$ that lies in the first quadrant. What is the volume of the solid generated by revolving R about the line $x + y = 2$?

(a) $\dfrac{\pi}{2\sqrt{2}}$

(b) $\dfrac{\pi}{2}$

(c) $\dfrac{\pi\sqrt{2}}{3}$

(d) $\dfrac{\pi^2}{2}$

(e) $\dfrac{\pi^2}{3\sqrt{2}}$

(f) $\dfrac{\pi^2}{4}$

(g) $\dfrac{\pi^2\sqrt{2}}{3}$

(h) $\dfrac{\pi^2}{2\sqrt{2}}$

$\boxed{\text{(i)}}$ None of the above

5. The series $\displaystyle\sum_{n=2}^{\infty} \frac{3^n}{n!}$ converges to

(a) $\ln 3$

(b) $\ln 2$

(c) $\ln(3) - 1$

(d) $\dfrac{3^{n+1}}{n+1}$

(e) ∞

(f) e^3

(g) $\cos 3$

$\boxed{\text{(h)}}$ $e^3 - 4$

(i) 3^e

6. The interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} n^2(7x-3)^n$ is

(a) $\left(-\dfrac{3}{7}, \dfrac{3}{7}\right)$

(b) $\left(-\dfrac{7}{3}, \dfrac{7}{3}\right)$

(c) $(-1, 1)$

(d) $(0, 1)$

(e) $\left(\dfrac{1}{7}, 1\right)$

$\boxed{\text{(f)}}$ $\left(\dfrac{2}{7}, \dfrac{4}{7}\right)$

(g) $(0, \infty)$

(h) $(-\infty, \infty)$

(i) None of these

7. The integral $\int_2^{e+1} (x-1)\ln(x-1)\,dx$ is equal to

(a) $\dfrac{e^2-1}{2}$

(b) e^2+1

(c) $\dfrac{e^2+1}{2}$

$\boxed{\text{(d)}}$ $\dfrac{e^2+1}{4}$

(e) $\dfrac{e^2-1}{4}$

(f) e^2-1

8. The graph of the polar equation $r = 2\cos(n\theta)$ has how many petals?

(a) n petals if n is even, $2n$ petals if n is odd

(b) $n/2$ petals if n is odd, n petals if n is even

$\boxed{\text{(c)}}$ n petals if n is odd, $2n$ petals if n is even

(d) $2n$ petals

(e) n petals

(f) $n/2$ petals

(g) None of these

Part II: Written Solutions

For problems 9 – 18, write your answers in the space provided. Neatly show your work for full credit.

9. Evaluate each integral

(a) $\displaystyle\int \frac{dx}{2+x-x^2}$

$$\begin{aligned}\int \frac{dx}{2+x-x^2} &= \int \frac{1}{3}\ln(x+1) - \frac{1}{3}\ln(x-2)\,dx \\ &= \frac{1}{3}\ln\left|\frac{x+1}{x-2}\right| + C\end{aligned}$$

(b) $\displaystyle\int \sec^3(2x)\,dx$

Integration by partial fraction with $u = \sec 2x,\ dv = \sec^2 x\,dx$

$$\int \sec^3(2x)\,dx = \frac{1}{2}\sec 2x \tan 2x - \int \sec 2x \tan^2 2x\,dx$$

$$\int \sec 2x \tan^2 2x\,dx = \int \sec 2x(\sec^2 2x - 1)\,dx = \int \sec^3 2x\,dx - \int \sec 2x\,dx$$

Moving $\int \sec^3 2x$ to LHS,

$$\int \sec^3(2x)\,dx = \frac{1}{4}\tan(2x)\sec(2x) + \frac{1}{4}\ln(\sec(2x) + \tan(2x)) + C$$

10. Find the general solution, in the form $y = f(x)$, to the differential equation

$$\frac{dy}{dx} = (4 + y^2)(4 + x^2).$$

Separating variables and integrating

$$\frac{dy}{4+y^2} = \int (4 + x^2)\, dx$$

$$y = 2\tan\left(8x + \frac{2}{3}x^3 + C\right)$$

11. Find the length of the graph of $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$, on the interval $1 \le x \le 2$.

Length of graph is given by the arc length formula $\int_1^2 \sqrt{1 + (y')^2}\, dx$

Now, as $y' = \frac{1}{2}\left(x - \frac{1}{x}\right)$, so the length of graph is

$$\int_1^2 \sqrt{1 + \frac{1}{4}\left(x^2 - 2 + \frac{1}{x^2}\right)}\, dx = \int_1^2 \frac{1}{2}\left(x + \frac{1}{x}\right)\, dx = \frac{3}{4} + \frac{1}{2}\ln 2$$

12. Find the centroid of the region that lies within the first quadrant and is bounded above by $y = 1 - x^2$.

$$A = \int_0^1 (1 - x^2)\, dx = \frac{2}{3}$$

$$m_y = \int_0^1 x(1 - x^2)\, dx = \frac{1}{4}$$

$$m_x = \frac{1}{2}\int_0^1 (1 - x^2)^2\, dx = \frac{4}{15}$$

$$\bar{x} = \frac{m_y}{A} = \frac{3}{8}$$

$$\bar{y} = \frac{m_x}{A} = \frac{2}{5}$$

13. Find the area enclosed by the polar curves $r = 2 - \cos\theta$ and $r = 1$.

Unit circle lies entirely inside the first curve,

Area is

$$\int_0^{2\pi} \frac{1}{2}r^2(\theta)\, d\theta - \pi\,(1)^2 = \int_0^{2\pi} \frac{1}{2}(2 - \cos\theta)^2\, d\theta - \pi = \frac{7}{2}\pi$$

14. Use the first three non-zero terms of the MacLaurin series for e^{-x^2} to estimate the definite integral $\int_0^2 e^{-x^2}\,dx$. Write your answer as a fraction, if possible.

$$e^{-x^2} = 1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + ...$$

$$\begin{aligned}\int_0^2 e^{-x^2}\,dx &= \int_0^2 \left(1 - x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + ...\right)dx \\ &\approx \int_0^2 1 - x^2 + \frac{1}{2!}x^4\,dx \\ &= x - \frac{x^3}{3} + \frac{x^5}{10}\Big|_0^2 = \frac{38}{15}\end{aligned}$$

15. Find the mass of the circular region $x^2 + y^2 \le 1$, whose density at each point is twice the distance from the point to the origin.

Mass is given by

$$\int_0^1 2r \cdot 2\pi r\,dr = \frac{4\pi r^3}{3}\Big|_0^1 = \frac{4\pi}{3}$$

16. Find the sum of the power series $\sum_{n=1}^{\infty} nx^{n-1}$ (as a rational function of x).

The power series $\sum_{n=1}^{\infty} nx^{n-1}$ is obtained by differentiating $f(x) = \sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$, so

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}.$$

17. Determine whether each of the following infinite series converges. State any convergence/divergence test you used.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Converges by comparison with $1/n^2$.

(b) $\sum_{n=1}^{\infty} \frac{e^n}{n^{30}+2^n}$

Diverges by divergence test (terms not approaching 0)

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$

Converges by alternate series test

18. Find the definite integral $\int_0^1 x^3\sqrt{1-x^2}\,dx$.

Let $y^2 = 1 - x^2$, then

$$\int_0^1 x^3\sqrt{1-x^2}\,dx = \int_0^1 (1-y^2)y^2\,dy = \frac{2}{15}$$