Math 113 (Calculus II)
Final Exam Form A KEY

Multiple Choice. Fill in the answer to each problem on your scantron. Make sure your name, section and instructor is on your scantron.

1. Here is a series $\sum_{k=1}^{\infty} \frac{(-1)^n}{\sqrt{n(n^2+1)}}$. Which of the following is true?

   a) The series does not converge absolutely by the root test.

   b) The series diverges by the integral test.

   c) The series converges conditionally by the ratio test.

   d) The series converges conditionally.

   e) The series converges absolutely by a limit comparison test.

   f) The series converges absolutely by the ratio test.

   g) The series neither diverges nor converges.

   Solution: e)

2. Compute the sum

$$\sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k$$

   a) $\frac{5}{12}$ b) $\frac{3}{2}$ c) 1

   d) $\frac{4}{5}$ e) $\frac{3}{4}$ f) 2

   g) The series diverges by the root test.

3. Find $\int_0^{\pi/4} \tan^2(x)\sec^2(x)\,dx$.

a) $\frac{\pi}{4}$ b) $\frac{1}{3}$ c) $\frac{1}{4}$

d) $\frac{\pi}{3}$ e) $\pi - 1$ f) π

g) None of the above.

4. A parameterization of the part of the hyperbola $x^2 - \frac{y^2}{4} = 1$ for $x > 0$ is

a) $x = \cosh(t), y = \sinh(2t), t \in \mathbb{R}$ b) $x = \cosh(t), y = 2\sinh(t), t \in \mathbb{R}$

c) $x = \cosh(t), y = 2\sinh(2t), t \in \mathbb{R}$ d) $x = \sinh(t), y = \sinh(2t), t \in \mathbb{R}$

e) None of the above.

5. Find $\int_0^1 x^3 \cos\left(x^2\right) dx$

a) $\frac{1}{2}\sin(1) - \frac{1}{2}\cos 1$ b) $\frac{3}{4}\sin(1) - \frac{1}{2}\cos(1)$ c) $\frac{1}{2}\cos 1 + \frac{1}{2}\sin 1 - \frac{1}{2}$

d) $\frac{1}{6} - \frac{1}{6}\sin(1)$ e) $\frac{1}{6} - \frac{1}{6}\sin(1) + C$ f) None of the above.

6. Find the first four terms of the power series for $(1+2x)^{1/2}$

a) $1 + 3x - \frac{4}{3}x^2 + \frac{1}{81}x^3$ b) $1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3$

c) $1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3$ d) $1 + \frac{1}{3}x - \frac{1}{6}x^2 + \frac{1}{27}x^3$

e) None of the above.

7. Find the interval of convergence of the power series

$$\sum_{k=1}^{\infty} 3^k \frac{x^k}{k^2}.$$

a) $\left(-\frac{1}{3}, \frac{1}{3}\right)$

b) $(0, \frac{1}{3})$

c) $(-\frac{1}{3}, 0)$

d) $\left[-\frac{1}{3}, \frac{1}{3}\right]$

e) $[-\frac{1}{3}, \frac{1}{3})$

f) $(-3, 3)$

g) The series converges for all values of x.

8. Find the area between the graphs of $y = \sin(x)$ and $y = \cos(x)$ for $x \in [0, \pi/2]$.

a) $\frac{4}{3}\pi - \frac{1}{3}$

b) $-2 + 2\sqrt{2}$

c) 0

d) 2

e) None of the above.

Short Answer. Fill in the blank with the appropriate answer. 2 points each

9. (20 points)

(a) What is the correct substitution to use in computing the integral, $\int_0^1 \sqrt{1+x^2}dx$? $\underline{x = \tan(u)}$

(b) Find $\lim_{n\to\infty} n \ln\left(1+2n^{-1}\right)$ $\underline{\ 2\ }$

(c) Find the first 3 nonzero terms of the power series of $\sin\left(x^2\right)$ centered at 0.
$\underline{x^2 - \frac{1}{6}x^6 + \frac{1}{120}x^{10}}$

(d) Let $f(x) = \cos(x)$. Find the first three terms of the power series of f centered at $\pi/4$.
$\underline{\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x - \frac{\pi}{4}) - \frac{\sqrt{2}}{4}(x - \frac{\pi}{4})^2}$

(e) What number equals $\sum_{k=2}^{\infty} \frac{1}{3^k}$? $\underline{\ \frac{1}{6}\ }$

(f) In the integral $\int_0^1 \left(1+x^5\right)^{1/2} dx$ the substitution, $x = \sin(u)$ is used. Write the integral

which results. Do not try to work the integral. $\underline{\int_0^{\pi/2} (1+\sin^5 u)^{1/2} \cos u \, du}$

(g) Find the antiderivative, $\int 3x^2 \sec^2\left(x^3\right) dx$. $\underline{\tan\left(x^3\right) + C}$

(h) What is the formula for the arc length of the graph of the function $y = f(x)$ for

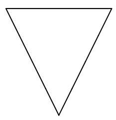
$x \in [a, b]$? $\underline{\int_a^b \sqrt{1 + (f'(x))^2}\, dx}$

(i) What is $\lim_{n\to\infty} n^2 \tan\left(\frac{1}{4n^2}\right)$? $\underline{\ \frac{1}{4}\ }$

(j) Identify $\int \sec(2x)\, dx$. $\underline{\frac{1}{2} \ln|\sec 2x + \tan 2x| + C}$

Free response: Give your answer in the space provided. Answers not placed in this space will be ignored. 6 points each

10. (6 points) A circular cone having side view shown below having radius 5 feet and height 10 feet is full of a fluid which weighs $1/\pi$ pounds per cubic foot. Find the work needed to pump this fluid out the top of the tank.



Solution:

Version A:

$$\int_0^{10} \frac{1}{\pi}h \cdot \pi(5 - \frac{h}{2})^2\, dh = \int_0^{10} (25h - 5h^2 + \frac{1}{4}h^3)\, dh$$

or

$$\int_0^{10} \frac{1}{\pi}(10 - h) \cdot \pi(\frac{h}{2})^2\, dh = \int_0^{10} (\frac{5}{2}h^2 - \frac{1}{4}h^3)\, dh$$

$$= \frac{625}{3}.$$

Version B:

$$\int_0^{6} \frac{1}{\pi}h \cdot \pi(3 - \frac{h}{2})^2\, dh = \int_0^{6} (9h - 3h^2 + \frac{1}{4}h^3)\, dh$$

or

$$\int_0^{6} \frac{1}{\pi}(6 - h) \cdot \pi(\frac{h}{2})^2\, dh = \int_0^{6} (\frac{3}{2}h^2 - \frac{1}{4}h^3)\, dh$$

$$= 27.$$

11. (6 points) Find the area enclosed by the curve $r = 2 + \cos\theta$ for $\theta \in [0, 2\pi]$.

Solution:

Version A:

$$\frac{1}{2}\int_0^{2\pi} (2+\cos\theta)^2\, d\theta = \frac{1}{2}\int_0^{2\pi} (4 + 4\cos\theta + \cos^2\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (4 + 4\cos\theta + \frac{1}{2} + \frac{1}{2}\cos(2\theta))\, d\theta = \int_0^{2\pi} (\frac{9}{4} + 2\cos\theta + \frac{1}{4}\cos(2\theta))\, d\theta$$

$$= \left(\frac{9}{4}\theta + 2\sin\theta + \frac{1}{8}\sin(2\theta)\right)_0^{2\pi}$$

$$= \frac{9}{2}\pi$$

Note: Students can also use a reduction formula or integration by parts to integrate $\cos^2\theta$.

Version B:

$$\frac{1}{2}\int_0^{2\pi} (3+\cos\theta)^2\, d\theta = \frac{1}{2}\int_0^{2\pi} (9 + 6\cos\theta + \cos^2\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (9 + 6\cos\theta + \frac{1}{2} + \frac{1}{2}\cos(2\theta))\, d\theta = \int_0^{2\pi} (\frac{19}{4} + 3\cos\theta + \frac{1}{4}\cos(2\theta))\, d\theta$$

$$= \left(\frac{19}{4}\theta + 3\sin\theta + \frac{1}{8}\sin(2\theta)\right)_0^{2\pi}$$

$$= \frac{19}{2}\pi$$

12. (6 points) Find

(a) $\displaystyle\int \frac{4x+11}{(x+4)(x-1)} dx$

Solution:

$$\frac{4x+11}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}$$
$$4x + 11 = A(x-1) + B(x+4).$$

If $x = -4$, then the above becomes $-5 = -5A$, so $A = 1$.
If $x = 1$, then the above becomes $15 = 5B$, so $B = 3$.

$$\int \frac{4x+11}{(x+4)(x-1)} = \int \left(\frac{1}{x+4} + \frac{3}{x-1}\right) dx$$
$$= \ln|x+4| + 3\ln|x-1| + C.$$

Note:

$$\ln\left|\frac{x+4}{(x-1)^3}\right| + C$$

is also acceptable.

(b) $\int \sqrt{3-x^2}dx$

Solution:

Version A:

Let $\sqrt{3}\sin\theta = x$. Then, $dx = \sqrt{3}\cos\theta$, and

$$\int \sqrt{3-x^2}dx = \int \sqrt{3-3\sin^2\theta}\sqrt{3}\cos\theta\, d\theta = 3\int \cos^2\theta\, d\theta$$

$$= \frac{3}{2}\int (1+\cos(2\theta))\, d\theta = \frac{3}{2}(\theta + \frac{1}{2}\sin(2\theta)) + C = \frac{3}{2}\theta + \frac{3}{2}\sin\theta\cos\theta + C.$$

Since $\sin\theta = \frac{x}{\sqrt{3}}$, we can populate a right triangle with x as the opposite side, and $\sqrt{3}$ as the hypotenuse. Thus, $\sqrt{3-x^2}$ will be the adjacent side, and

$$\cos\theta = \frac{\sqrt{3-x^2}}{\sqrt{3}}.$$

Thus, the integral is

$$\frac{3}{2}\sin^{-1}\left(\frac{x}{\sqrt{3}}\right) + \frac{1}{2}x\sqrt{3-x^2} + C.$$

Version B:

Let $\sqrt{5}\sin\theta = x$. Then, $dx = \sqrt{5}\cos\theta$, and

$$\int \sqrt{5-x^2}dx = \int \sqrt{5-5\sin^2\theta}\sqrt{5}\cos\theta\, d\theta = 5\int \cos^2\theta\, d\theta$$

$$= \frac{5}{2}\int (1+\cos(2\theta))\, d\theta = \frac{5}{2}(\theta + \frac{1}{2}\sin(2\theta)) + C = \frac{5}{2}\theta + \frac{5}{2}\sin\theta\cos\theta + C.$$

Since $\sin\theta = \frac{x}{\sqrt{5}}$, we can populate a right triangle with x as the opposite side, and $\sqrt{5}$ as the hypotenuse. Thus, $\sqrt{5-x^2}$ will be the adjacent side, and

$$\cos\theta = \frac{\sqrt{5-x^2}}{\sqrt{5}}.$$

Thus, the integral is

$$\frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right) + \frac{1}{2}x\sqrt{5-x^2} + C.$$

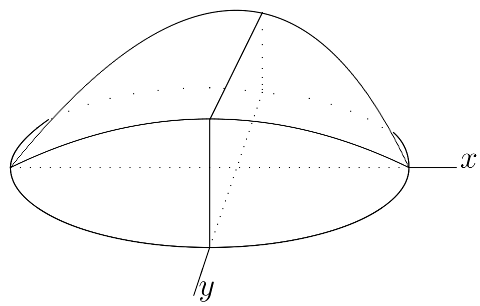
13. (6 points) The region between $y = \sin x$ which lies between $x = 0$, $x = \pi/2$, and the x axis is revolved about the line $x = 0$. Find the volume of the resulting solid of revolution.

Solution:

$$\int_0^{\pi/2} 2\pi x\sin(x)\, dx = 2\pi(-x\cos x + \sin x)_0^{\pi/2}$$

$$= 2\pi$$

14. (6 points) The base of a solid is the inside of the circle, $x^2+y^2 \leq 4$. Cross sections perpendicular to the x axis are squares the length of a side corresponding to x being equal to the width of the base at that value of x. Find the volume of the resulting solid. A sketch of one such solid is shown.



Solution:

Version A:

The vertical cross section of a circle of radius 2 is $2\sqrt{4-x^2}$. Thus, the volume is

$$\int_{-2}^{2} (2\sqrt{4-x^2})^2\, dx = \int_{-2}^{2} (16-4x^2)\, dx = \left(16x - \frac{4}{3}x^3\right)_{-2}^{2}$$

$$= \frac{128}{3}$$

Version B:

The vertical cross section of a circle of radius 3 is $2\sqrt{9-x^2}$. Thus, the volume is

$$\int_{-3}^{3} (2\sqrt{9-x^2})^2\, dx = \int_{-3}^{3} (36-4x^2)\, dx = \left(36x - \frac{4}{3}x^3\right)_{-2}^{2}$$

$$= 144$$

15. (6 points) Determine whether the following series converges and explain your answer.

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{\sqrt{n^3+5}}$$

Solution:

Version A:

Notice that

$$\left|\frac{\sin(n)}{\sqrt{n^3+5}}\right| \le \frac{1}{\sqrt{n^3+5}} \le \frac{1}{n^{3/2}}.$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$$

converges (p series, $p > 1$), the above series converges absolutely by use of the comparison test.

This can be proven also by using the comparison test and then the limit comparison test.

Version B:

Notice that

$$\left|\frac{\sin(n)}{\sqrt{n^5+5}}\right| \le \frac{1}{\sqrt{n^5+5}} \le \frac{1}{n^{5/2}}.$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$$

converges (p series, $p > 1$), the above series converges absolutely by use of the comparison test.

16. (6 points) Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} 2^n \sqrt{n} x^n.$$

Solution:

Version A: Using the Ratio test,

$$\frac{2^{n+1}\sqrt{n+1}\,|x|^{n+1}}{2^n\sqrt{n}\,|x|^n} = 2\frac{\sqrt{n+1}}{\sqrt{n}}\,|x| \to 2\,|x|\,.$$

Setting $2\,|x| < 1$, we have $|x| < 1/2$. Thus, the radius of convergence is $1/2$.

Version B:

Similar, except the radius of convergence is $1/5$.

17. (6 points) Find $\int_0^\infty x^2 e^{-x} dx$

Solution: Use integration by parts. $u = x^2$, $du = 2x\, dx$, $dv = e^{-x}\, dx$, $v = -e^{-x}$.

$$\int_0^\infty x^2 e^{-x} dx = \lim_{b\to\infty} \int_0^b x^2 e^{-x} dx.$$

$$\int_0^b x^2 e^{-x} dx = -x^2 e^{-x}|_0^b + 2\int_0^b x e^{-x}\, dx.$$

For the second integral, we use parts again:
$u = x$, $du = dx$, $dv = e^{-x}\, dx$, $v = -e^{-x}$. Thus,

$$\int_0^b x e^{-x}\, dx = -xe^{-x}|_0^b + \int_0^b e^{-x}\, dx = (-xe^{-x} - e^{-x})_0^b = 1 - be^{-b} - e^{-b}$$

and

$$\int_0^b x^2 e^{-x} dx = 2 - b^2 e^{-b} - 2be^{-b} - 2e^{-b}.$$

Thus,

$$\int_0^\infty x^2 e^{-x} dx = \lim_{b\to\infty} (2 - b^2 e^{-b} - 2be^{-b} - 2e^{-b}) = 2.$$

Note: Students know very well by now that $b^2 e^{-b}$ and be^{-b} go to 0 as $b \to \infty$. Most will not try to prove it.

END OF EXAM