Math 113 (Calculus II)
Final Exam – Form A – Fall 2012 RED KEY

Part I: Multiple Choice *Mark the correct answer on the bubble sheet provided.*

1. Which of the following series converge absolutely?

$$1)\ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \qquad 2)\ \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \qquad 3)\ \sum_{n=1}^{\infty} \frac{1}{n^3}$$

a) None	b) 1	c) 2
d) 3	e) 1, 2	f) 1, 3
g) 2, 3	h) 1, 2, 3	

Solution: f)

2. Find the radius of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!(x-2)^n}{n^2 3^n}$.

a) 0	b) 1	c) 2
d) 3	e) 2/3	f) 4/3
g) 1/3	h) -2	i) -3
j) ∞		

Solution: a)

3. What is the coefficient of x^{100} in the Maclaurin series of e^{-3x^2}?

a) 0	b) 3^{100}	c) $\dfrac{3^{100}}{100}$
d) $-\dfrac{3^{100}}{100!}$	e) $-\dfrac{3^{25}}{25!}$	f) $\dfrac{3^{50}}{50}$
g) $\dfrac{3^{50}}{50!}$	h) $-\dfrac{3^{50}}{50!}$	i) Diverges

Solution: g)

Solution: $e^{-3x^2} = \displaystyle\sum_{k=0}^{\infty} \frac{(-3x^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-3)^k x^{2k}}{k!}$. When $k = 50$, we obtain the term containing x^{100}. The coefficient is $\dfrac{(-3)^{50}}{50!} = \dfrac{3^{50}}{50!}$.

4. When a particle is located a distance x feet from the origin, a force of $x^2 + 2x$ pounds acts on it. How much work (measured in foot-pounds) is done in moving it from $x = 1$ to $x = 3$?

a) 6	b) 9	c) 50/3
d) 17/3	e) −16/3	f) −45/2
g) 27/2	h) 15/2	i) 0

Solution: c)

Solution: $W = \int_1^3 (x^2+2x)\,dx = (x^3/3+x^2)\big|_1^3 = (9+9)-(1/3+1) = 17-1/3 = \frac{51-1}{3} = 50/3$

5. The region between the curve $y = \frac{1}{x^p}$ and the x-axis for $0 < x \leq 1$ is rotated about the x-axis to form a solid of revolution. For which positive values of p does this solid have *finite* volume?

a) $0 < p$	b) $0 < p < 1$	c) $1 < p$
d) $0 < p < 1/2$	e) $1/2 < p$	f) $0 < p < 2$
g) $2 < p$	h) The volume is infinite for all positive p.	

Solution: d)

Solution: The volume is $V = \pi \int_0^1 \left(\frac{1}{x^p}\right)^2 dx = \pi \int_0^1 \frac{dx}{x^{2p}}$, where the integral is improper since $p > 0$. The integral converges when $p < 1/2$ but diverges when $p \geq 1/2$.

6. Evaluate the integral $\int_0^{\pi/2} \sin^3(x)\cos^2(x)\,dx$.

a) 1/15	b) 2/15	c) 3/2
d) 1/3	e) 3/5	f) π
g) $\pi/3$	h) $\pi/2$	i) 0

Solution: b)

Solution: Let $u = \cos x$, $du = -\sin x\,dx$. Then

$$\int_0^{\pi/2} \sin^3(x)\cos^2(x)\,dx = \int_0^{\pi/2} (1-\cos^2 x)\cos^2 x \sin x\,dx = \int_1^0 (1-u^2)u^2(-du)$$
$$= \int_0^1 (u^2 - u^4)\,du = \tfrac{1}{3} - \tfrac{1}{5} = \tfrac{2}{15}.$$

7. Evaluate the integral $\int_{-\infty}^{\infty} \frac{dx}{1+4x^2}$.

a) 0 b) 1 c) $\arcsin(1/2)$

d) π e) 2π f) $\pi/2$

g) $\arctan(\pi)$ h) $\sqrt{2}$ i) $\dfrac{1}{\sqrt{2}}$

Solution: f)

Solution: Set $u = 2x$ and $du = 2\,dx$. Then $dx = du/2$ and

$$\int_{-\infty}^{\infty} \frac{dx}{1+4x^2} = \int_{-\infty}^{\infty} \frac{du/2}{1+u^2} = \tfrac{1}{2}\arctan(u)\Big|_{-\infty}^{\infty} = \tfrac{1}{2}(\pi/2) - \tfrac{1}{2}(-\pi/2) = \pi/2.$$

8. Which integral represents the length of the curve $y = \sin x + \cos x$, $0 \le x \le \pi/4$? (You might need to set up an integral and do a short calculation.)

a) $\displaystyle\int_0^{\pi/4} \sqrt{2 + 2\cos x \sin x}\, dx$ b) $\displaystyle\int_0^{\pi/4} \sqrt{2 - 2\cos x \sin x}\, dx$

c) $\displaystyle\int_0^{\pi/4} \sqrt{2 - \cos x \sin x}\, dx$ d) $\displaystyle\int_0^{\pi/4} \sqrt{1 - 2\cos x \sin x}\, dx$

e) $\displaystyle\int_0^{\pi/4} \sqrt{1 - \cos x \sin x}\, dx$ f) $\displaystyle\int_0^{\pi/4} \sqrt{1 + \cos x \sin x}\, dx$

Solution: b)

Solution: If $f(x) = \sin x + \cos x$, the length of the curve is

$$\int_0^{\pi/4} \sqrt{1 + [f'(x)]^2}\, dx = \int_0^{\pi/4} \sqrt{1 + [\cos x - \sin x]^2}\, dx = \int_0^{\pi/4} \sqrt{1 + \cos^2 x - 2\cos x \sin x + \sin^2 x}\, dx$$
$$= \int_0^{\pi/4} \sqrt{2 - 2\cos x \sin x}\, dx$$

9. Which integral represents the area of the surface obtained by rotating the curve

$$y = e^x, \quad 1 \le y \le 8$$

about the y-axis.

a) $\displaystyle\int_1^{8} 2\pi x\sqrt{1 + e^{2x}}\, dx$ b) $\displaystyle\int_0^{\ln 8} 2\pi\sqrt{1 + e^{2x}}\, dx$

c) $\displaystyle\int_0^{\ln 8} 2\pi x\sqrt{1 + e^{2x}}\, dx$ d) $\displaystyle\int_0^{\ln 8} 2\pi e^x\sqrt{1 + e^{2x}}\, dx$

e) $\displaystyle\int_1^{8} 2\pi e^x\sqrt{1 + e^{x}}\, dx$ f) $\displaystyle\int_1^{8} 2\pi e^x\sqrt{1 + e^{2x}}\, dx$

Solution: c)

Solution: For the curve $y = e^x$, $1 \le y \le 8$ the values of x satisfy $0 \le x \le \ln 8$. Since rotation is about the y-axis, the radius is x.

$$\int 2\pi x ds = \int_0^{\ln 8} 2\pi x \sqrt{1 + (dy/dx)^2}\, dx = \int_0^{\ln 8} 2\pi x \sqrt{1 + e^{2x}}\, dx$$

10. If $(\bar{x}, \bar{y})$ is the centroid of the region bounded by the line $y = x$ and the parabola $y = x^2$, what is $\bar{y}$?

a) 0	b) 1/2	c) 1/3
d) 2/3	e) 1/4	f) 3/4
g) 1/5	h) 2/5	i) 3/5

Solution: h)

Solution: The area is $A = \int_0^1 (x - x^2)\, dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$. Then $\bar{y} = \frac{1}{A} \int_0^1 \frac{1}{2}[(x)^2 - (x^2)^2]dx = (1/2)/(1/6)[\frac{1}{3} - \frac{1}{5}] = 3(2/15) = 2/5$

11. A curve is parametrized by the equations $x = 6\sin t$ and $y = t^2 + t$. Find the slope of the line that is tangent to this curve at the point $(0, 0)$.

a) 0	b) 1	c) 1/2
d) 2	e) 1/3	f) 3
g) 1/6	h) 6	i) Undefined

Solution: g)

Solution:

$x = 6\sin t,\ y = t^2 + t;\ (0, 0)$.

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t+1}{6\cos t}$. The point $(0, 0)$ corresponds to $t = 0$, so the slope of the tangent at that point is $\frac{1}{6}$. An equation of the tangent is therefore $y - 0 = \frac{1}{6}(x - 0)$, or $y = \frac{1}{6}x$.

12. Determine the exact value of the geometric alternating series:

$$\frac{3}{7} - \frac{3}{7^2} + \frac{3}{7^3} - \frac{3}{7^4} + \cdots$$

a) 1/2	b) 3/4	c) 7/6
d) 7/4	e) 3/8	f) 7/8

Solution: e)

Solution: $\frac{3/7}{1 - (-1/7)} = \frac{3/7}{8/7} = 3/8$

13. Which of the following three tests will establish that the series $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+2)}$ converges?

1) Comparison Test with $\displaystyle\sum_{n=1}^{\infty} 2n^{-2}$

2) Limit Comparison Test with $\displaystyle\sum_{n=1}^{\infty} n^{-2}$

3) Comparison Test with $\displaystyle\sum_{n=1}^{\infty} 3n^{-2}$

a) None b) 1 c) 2

d) 3 e) 1, 2 f) 1, 3

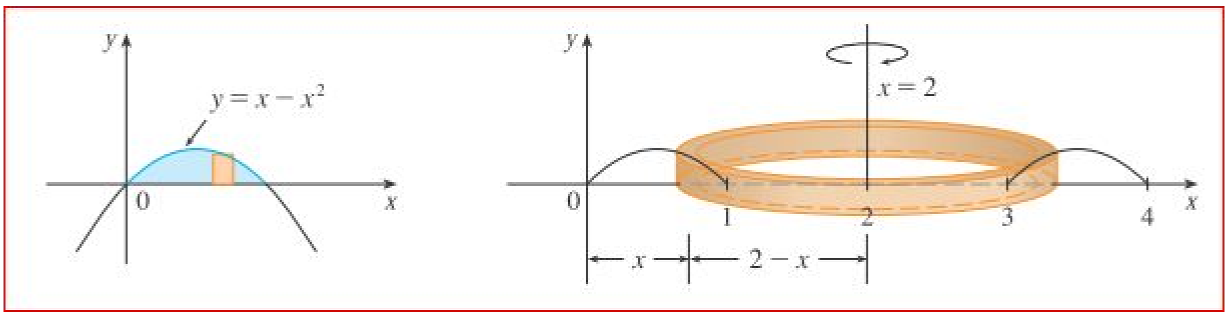
g) 2, 3 h) 1, 2, 3

Solution: g)

Part II: Written Response *Neatly write the solution to each problem. Complete explanations are required for full credit.*

14. (6 points) Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the vertical line $x = 2$.

Solution: We will compute volume by using cylindrical shells as in the figure below:



The cylindrical shell of radius $2 - x$ has height $x - x^2$. So,

$$
\begin{aligned}
V &= \int_0^1 2\pi \cdot \text{radius} \cdot \text{height}\, dx = 2\pi \int_0^1 (2-x)(x-x^2)\, dx \\
&= 2\pi \int_0^1 (x^3 - 3x^2 + 2x)\, dx \\
&= 2\pi \left(\frac{x^4}{4} - x^3 + x^2 \right) \Bigg|_0^1 = 2\pi \left(\frac{1^4}{4} - 1^3 + 1^2 \right) - 2\pi \left(\frac{0}{4} - 0^3 + 0^2 \right) \\
&= \boxed{\frac{\pi}{2}}
\end{aligned}
$$

15. (6 points) Evaluate $\displaystyle\int x\sin(3x)\,dx$.

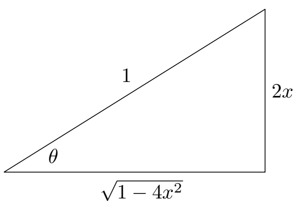
Solution: Use integration-by-parts: $\displaystyle\int u\,dv = uv - \int v\,du$.

$$\int x\sin(3x)\,dx = \underbrace{x}_{u}\underbrace{\left(-\frac{\cos(3x)}{3}\right)}_{v} - \int \underbrace{\left(-\frac{\cos(3x)}{3}\right)}_{v}\underbrace{dx}_{du} \quad \text{where} \begin{cases} u = x & dv = \sin(3x)\,dx \\ du = dx & v = -\frac{\cos(3x)}{3} \end{cases}$$

$$= -\frac{x\cos(3x)}{3} + \frac{\sin(3x)}{9} + C$$

16. (6 points) Evaluate $\displaystyle\int \sqrt{1-4x^2}\,dx$.

Solution: Use the substitution $x = \frac{1}{2}\sin\theta$ and $dx = \frac{1}{2}\cos\theta$ for $-\pi/2 < \theta < \pi/2$ with the associated trigonometric diagram:



$$\begin{aligned}\int \sqrt{1-4x^2}\,dx &= \int \sqrt{1-\sin^2\theta}\cdot\tfrac{1}{2}\cos\theta\,d\theta = \tfrac{1}{2}\int |\cos\theta|\cos\theta\,d\theta \\ &= \tfrac{1}{2}\int \cos^2\theta\,d\theta = \tfrac{1}{2}\int \tfrac{1}{2}(1+\cos(2\theta))\,d\theta \\ &= \tfrac{1}{4}\left(\theta + \tfrac{1}{2}\sin(2\theta)\right) + C \\ &= \tfrac{1}{4}\left(\theta + \sin\theta\cos\theta\right) + C \\ &= \tfrac{1}{4}\left(\arcsin(2x) + 2x\sqrt{1-4x^2}\right) + C \\ &= \boxed{\tfrac{1}{4}\arcsin(2x) + \tfrac{1}{2}x\sqrt{1-4x^2} + C}\end{aligned}$$

17. (6 points) Evaluate the integral $\displaystyle\int_1^\infty \frac{dx}{x^2+x}$.

Solution: First, find the partial fraction decomposition of the integrand:

$$\frac{1}{x^2+x} = \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \quad \Rightarrow \quad 1 = A(x+1) + Bx$$

Evaluating at $x = 0$ gives $A = 1$. Evaluating at $x = -1$ gives $B = -1$. So

$$\int \frac{dx}{x^2+x} = \int \left(\frac{1}{x} - \frac{1}{x+1}\right)dx = \ln|x| - \ln|x+1| + C = \ln\left|\frac{x}{x+1}\right| + C$$

The definite improper integral is

$$\int_1^\infty \frac{dx}{x^2+x} = \lim_{B\to\infty}\int_1^B \frac{dx}{x^2+x} = \lim_{B\to\infty}\left(\ln\left|\frac{x}{x+1}\right|\Bigg|_1^B\right)$$
$$= \lim_{B\to\infty}\left(\ln\left(\frac{B}{B+1}\right) - \ln\left(\frac{1}{1+1}\right)\right)$$
$$= \ln(1) - \ln\left(\frac{1}{2}\right) = 0 + \ln(2)$$
$$= \boxed{\ln(2)}$$

18. (6 points) Let $s(n) = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$. Find a large enough value of n such that $s(n) \geq 20$, and justify why this choice of n is large enough.

Hint: Think about the geometric reasoning used in the proof of the Integral Test.

Solution: Since $\frac{1}{\sqrt{k}}$ is a decreasing positive function of positive k, we regard the term $1/\sqrt{k}$ as a rectangle of height $1/\sqrt{k}$ and width 1 between k and $k+1$. Then we have

$$s(n) = \sum_{k=1}^{n}\frac{1}{\sqrt{k}} \geq \int_1^{n+1}\frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_1^{n+1} = 2\sqrt{n+1}-2.$$

We'll choose n large enough such that the lower bound for $s(n)$ is ≥ 20.

$$\begin{aligned} & 2\sqrt{n+1}-2 \geq 20 \\ \Leftrightarrow\quad & \sqrt{n+1}-1 \geq 10 \\ \Leftrightarrow\quad & \sqrt{n+1} \geq 11 \\ \Leftrightarrow\quad & n+1 \geq 121 \\ \Leftrightarrow\quad & n \geq 120 \end{aligned}$$

If we choose $n = 120$, then $s(n) \geq 20$.

[By doing a computer calculation, we find that $s(114) \approx 19.9406$ and $s(115) \approx 20.0338$. Thus, the smallest correct value of n would be $n = 115$. However, this would be difficult to check by hand. The upper bound in the integral above could be replaced with a smaller value like n resulting in a slightly cruder, but correct, lower bound.]

19. (6 points) Determine the interval of convergence for the power series $\sum_{n=1}^{\infty}\frac{(2x-1)^n}{5^n\sqrt{n}}$.

Solution: The series clearly converges if $x = 1/2$.

Assume $x \neq 1/2$. If $a_n = \frac{(2x-1)^n}{5^n\sqrt{n}}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2x-1)^{n+1}}{5^{n+1}\sqrt{n+1}}\cdot\frac{5^n\sqrt{n}}{(2x-1)^n}\right| = \lim_{n\to\infty}\frac{|2x-1|}{5}\sqrt{\frac{n}{n+1}} = \frac{|2x-1|}{5}$$

Then

$$\frac{|2x-1|}{5} < 1 \quad \Leftrightarrow \quad -5 < 2x-1 < 5 \quad \Leftrightarrow \quad -2 < x < 3.$$

By the Ratio Test the series converges for $x \in (-2,3)$ and diverges for $x < -2$ or $x > 3$.

If $x = -2$,

$$\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

is a convergent alternating series.

If $x = 3$,

$$\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

diverges by the p-series test with $p = 1/2 \leq 1$.

The interval of convergence is

$$\boxed{[-2,3)}$$

20. (6 points) Assuming $0 < x < 1$, evaluate the definite integral $\int_0^x \frac{du}{1+u^7}$ as a power series. Express the answer using summation notation.

Solution: In the integrand, $|u| < 1$. Hence we may use the formula for geometric series $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$ where $r = -u^7$.

$$\begin{aligned}\int_0^x \frac{du}{1+u^7} &= \int_0^x \left(\sum_{n=0}^{\infty}(-1)^n u^{7n}\right) du = \sum_{n=0}^{\infty}(-1)^n \int_0^x u^{7n}\,du \\ &= \sum_{n=0}^{\infty}(-1)^n \left(\left.\frac{u^{7n+1}}{7n+1}\right|_0^x\right) = \sum_{n=0}^{\infty}(-1)^n\left(\frac{x^{7n+1}}{7n+1} - \frac{0^{7n+1}}{7n+1}\right) \\ &= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{7n+1}}{7n+1}}\end{aligned}$$

21. (7 points) Find the Taylor series for the function $f(x) = \sqrt{x}$ centered at the value $a = 1$. Express the answer using summation notation.

Solution: [First Solution] Use the theorem on Binomial Series: In this case, for $|x-1| < 1$, we have

$$\sqrt{x} = \sqrt{1+(x-1)} = \boxed{\sum_{n=0}^{\infty}\binom{1/2}{n}(x-1)^n}$$

where, for real r, $\binom{r}{n} = \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}$.

[The textbook *does* use the notation $\binom{r}{0} = 1$, which is consistent with the traditional notational convention that an empty product equals 1.]

Solution: [Second Solution]

$$\begin{aligned}
f(x) &= x^{1/2} \\
f'(x) &= \tfrac{1}{2}x^{1/2-1} \\
f''(x) &= \tfrac{1}{2}(\tfrac{1}{2}-1)x^{1/2-2} \\
&\cdots \\
f^{(n)}(x) &= \tfrac{1}{2}(\tfrac{1}{2}-1)(\tfrac{1}{2}-2)\cdots(\tfrac{1}{2}-n+1)x^{1/2-n}
\end{aligned}$$

Then $f(1) = 1$ and for $n \geq 1$

$$f^{(n)}(1) = \tfrac{1}{2}(\tfrac{1}{2}-1)(\tfrac{1}{2}-2)\cdots(\tfrac{1}{2}-n+1)$$

The Taylor series centered at $a = 1$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} = \boxed{1 + \sum_{n=1}^{\infty} \frac{\tfrac{1}{2}(\tfrac{1}{2}-1)(\tfrac{1}{2}-2)\cdots(\tfrac{1}{2}-n+1)}{n!}(x-1)^n}$$

In a (mostly futile) attempt at simplification, we could manipulate the numerator in the previous formula as follows:

$$\begin{aligned}
\tfrac{1}{2}(\tfrac{1}{2}-1)(\tfrac{1}{2}-2)\cdots(\tfrac{1}{2}-n+1) &= (\tfrac{1}{2})(-\tfrac{1}{2})(-\tfrac{3}{2})\cdots(-\tfrac{2n-3}{2}) \\
&= \frac{(-1)^{n-1}1\cdot 3\cdot 5\cdots(2n-3)}{2^n} \\
&= \frac{(-1)^{n-1}(2n-3)!}{2^n\, 2\cdot 4\cdot 6\cdots(2n-4)} \\
&= \frac{(-1)^{n-1}(2n-3)!}{2^{2n-2}(n-2)!}
\end{aligned}$$

So, in the interval $|x-1| < 1$, we also could write

$$\sqrt{x} = 1 - \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}(2n-3)!}{2^{2n-2}(n-2)!n!}(x-1)^n$$

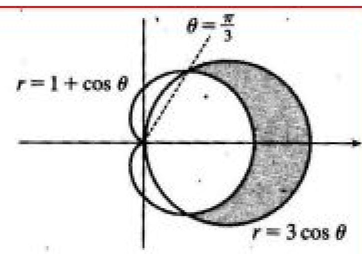
22. (6 points) Find the area of the region that lies inside the first curve and outside the second curve:

$$r = 3\cos\theta, \qquad r = 1 + \cos\theta.$$

Solution:

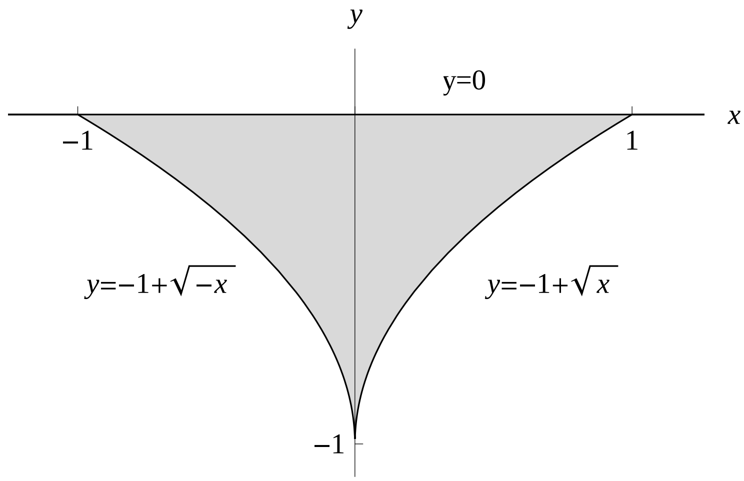
$3\cos\theta = 1 + \cos\theta \quad \Leftrightarrow \quad \cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3} \text{ or } -\frac{\pi}{3}.$

$$\begin{aligned}
A &= 2\int_0^{\pi/3} \tfrac{1}{2}[(3\cos\theta)^2 - (1+\cos\theta)^2]\,d\theta \\
&= \int_0^{\pi/3}(8\cos^2\theta - 2\cos\theta - 1)\,d\theta = \int_0^{\pi/3}[4(1+\cos 2\theta) - 2\cos\theta - 1]\,d\theta \\
&= \int_0^{\pi/3}(3 + 4\cos 2\theta - 2\cos\theta)\,d\theta = \left[3\theta + 2\sin 2\theta - 2\sin\theta\right]_0^{\pi/3} \\
&= \pi + \sqrt{3} - \sqrt{3} = \pi
\end{aligned}$$



23. (6 points) A trough is full of water. Its end is shaped like the shaded region in the picture. The boundaries of the region are the curves $y = 0$ and $y = -1 + |x|^{1/2}$ for $-1 \le x \le 1$. If the pressure at depth d is $P = \delta d$, where δ is a constant and d is measured in meters, set up a definite integral for the hydrostatic force F against the end of the trough.

[Note: Set up an integral for F, but don't evaluate the integral. The answer will involve δ.]



Solution: Solving for x in terms of y for the left and right boundary curves gives:

$$y = -1 + (-x)^{1/2} \quad \Rightarrow \quad x = -(1+y)^2$$
$$y = -1 + x^{1/2} \quad \Rightarrow \quad x = (1+y)^2$$

A thin horizontal strip at position y, where $-1 \le y \le 0$, is at depth $d = |y| = -y$. The width of this strip is $(1+y)^2 - [-(1+y)^2] = 2(1+y^2)$. Then

$$F = \int_{-1}^{0} \delta d\, dy = \int_{-1}^{0} \delta(-y)2(1+y)^2\, dy = \boxed{-2\delta \int_{-1}^{0} y(1+y)^2\, dy}$$

END OF EXAM