Name________________________________

Student Number________________________

Section Number________________________

Instructor____________________________

Math 113 – Winter 2005 – Key

Departmental Final Exam

Instructions:

- The time limit is 3 hours.
- Problem 1 consists of 13 short answer questions.
- Problems 2 through 9 are multiple choice questions.
- For problems 10 through 18 give the best answer and *justify* it with suitable reasons and/or relevant work.
- Work on scratch paper will not be graded.
- Do not show your work for problem 1.
- Please write neatly.
- Notes, books, and calculators are not allowed.
- Expressions such as $\ln(1)$, e^0, $\sin(\pi/2)$, etc. must be simplified for full credit.

For administrative use only:

1	/13
M.C.	/24
10	/7
11	/7
12	/7
13	/7
14	/7
15	/7
16	/7
17	/7
18	/7
Total	/100

Math 113 – Winter 2005

Departmental Final Exam

Part I: Short Answer and Multiple Choice Questions

Do not show your work for problem 1.

1. Fill in the blanks with the correct answer.

(a) The integral $\int_0^{\pi/2} \cos(2x)\,dx$ equals $\underline{\tfrac{1}{2}\sin 2x\big|_0^{\pi/2} = 0}$

(b) The integral $\int \sin x \cos^2 x\,dx$ equals $\underline{-\dfrac{1}{3}\cos^3 x + C}$

(c) The integral $\int_0^{\infty} \dfrac{dx}{1+x^2}$ equals $\underline{\tan^{-1} x\big|_0^{\infty} = \dfrac{\pi}{2}}$

(d) The radius of convergence of $\sum_{n=0}^{\infty} 2^n x^n$ is $\underline{\dfrac{1}{2}}$

(e) The first three lowest order terms of the power series of $(1+x)^{1/2}$ may be written as

$\underline{1 + \dfrac{1}{2}x + \dfrac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}{2!}x^2 + \cdots = 1 + \dfrac{1}{2}x - \dfrac{1}{8}x^2 + \cdots}$

(f) For what values of p does the following improper integral converge?

$\int_1^{\infty} \dfrac{dx}{x^p}$ $\underline{-p+1 < 0 \Rightarrow p > 1}$

(g) Indicate which convergence test one could use in determining the convergence/ divergence of

i. $\sum_{n=2}^{\infty} \dfrac{1}{n - \sqrt{n}}$ limit comparison with $1/n$, integral test, comparison test with $1/n$

ii. $\sum_{n=1}^{\infty} \dfrac{n}{n^3+1}$ comparison test/ limit comparison with $1/n^2$

iii. $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n!}$ ratio test, alternating series test, comparison test with $1/n!$

(h) State the nth term of the MacLaurin series for

i. e^x $\underline{\dfrac{1}{n!}x^n}$

ii. $\dfrac{1}{1-x}$ $\underline{x^n}$

(i) Express in terms of a quotient of integrals the y coordinate of the centroid of the region below $y = f(x)$ with $f(x) > 0$ for all x over $[-2, 2]$.

$\underline{\bar{y} = \int_{-2}^{2} \tfrac{1}{2}[f(x)]^2\,dx / \int_{-2}^{2} f(x)\,dx}$

(j) A focus of the hyperbola $\dfrac{(x-2)^2}{1} - \dfrac{(y-1)^2}{3} = 1$ is $\underline{[0,1] \text{ or } [4,1]}$

Problems 2 through 9 are multiple choice. Each multiple choice problem is worth 3 points. In the grid below fill in the square corresponding to each correct answer.

2	[A]	[B]	[C]	[D]	E	[F]	[G]	[H]	[I]
3	[A]	[B]	[C]	D	[E]	[F]	[G]	[H]	[I]
4	[A]	[B]	[C]	[D]	E	[F]	[G]	[H]	[I]
5	[A]	[B]	[C]	[D]	[E]	F	[G]	[H]	[I]
6	A	[B]	[C]	[D]	[E]	[F]	[G]	[H]	[I]
7	[A]	B	[C]	[D]	[E]	[F]	[G]	[H]	[I]
8	[A]	[B]	[C]	D	[E]	[F]	[G]	[H]	[I]
9	[A]	B	[C]	[D]	[E]	[F]	[G]	[H]	[I]

2. Find $\displaystyle\int_1^2 \frac{6+x^2+x}{(2+x)(4+x^2)}\,dx$

(a) $2\ln 2+\arctan 3-\ln 3+\frac{1}{2}\arctan 7-\frac{1}{2}\pi$

(b) $\ln 2+\frac{1}{8}\pi-\frac{1}{2}\arctan\frac{1}{2}$

(c) $3\ln 2+\frac{1}{8}\pi-\ln 3-\frac{1}{2}\arctan\frac{1}{2}$

(d) $2\ln 2+\arctan 2-\ln 3-\frac{1}{4}\pi$

(e) $2\ln 2+\frac{1}{8}\pi-\ln 3-\frac{1}{2}\arctan\frac{1}{2}$

(f) 0

(g) π

(h) None of the above

Answer: $2\ln 2+\frac{1}{8}\pi-\ln 3-\frac{1}{2}\arctan\frac{1}{2}$

3. The base of a solid is an elliptical region bounded by $\frac{1}{9}x^2+\frac{1}{4}y^2=1$, and cross sections perpendicular to the y axis are squares. Find the volume of the solid.

(a) 20

(b) 64

(c) 12

(d) 96

(e) 30

(f) 48

(g) 18

(h) 72

(i) None of the above

Answer: 96

4. Find the length of the graph of $y = x^{1/2} - \dfrac{x^{3/2}}{3}$ for $x \in [1, 6]$.

(a) $\sqrt{7} + 2\sqrt{6} - \frac{1}{3}$

(b) $\sqrt{6} + \frac{7}{3}\sqrt{7} - \frac{2}{3}$

(c) $\sqrt{6} + \frac{7}{3}\sqrt{7} - \frac{5}{3}$

(d) $\sqrt{6} + \frac{7}{3}\sqrt{7} - \frac{4}{3}$

(e) $3\sqrt{6} - \frac{4}{3}$

(f) $\sqrt{7} + \frac{16}{3}\sqrt{2} - \frac{4}{3}$

(g) $\sqrt{6} + \frac{85}{3}$

(h) None of the above

Answer: $3\sqrt{6} - \frac{4}{3}$

5. Find $\lim\limits_{x \to 0} \dfrac{2 - x^2 - 2\cos x}{x^4}$

(a) $-\frac{1}{14}$

(b) $-\frac{1}{6}$

(c) $-\frac{1}{3}$

(d) $\frac{1}{6}$

(e) $-\frac{1}{4}$

(f) $-\frac{1}{12}$

(g) $\frac{1}{14}$

(h) $-\frac{1}{10}$

(i) None of the above

Answer: $-\frac{1}{12}$

6. $\int_3^\infty e^{-t} \sin(4t)\, dt$

(a) $\frac{1}{17}e^{-3}(4\cos 12 + \sin 12)$

(b) $\frac{1}{17}e^{-3}(4\cos 12 - 1 + \sin 12)$

(c) $\frac{1}{17}e^{-3}(4\cos 12 - 3 + \sin 12)$

(d) $\frac{1}{17}e^{-3}(4\cos 12 - 2 + \sin 12)$

(e) $\frac{1}{16}e^{-3}(4\cos 12 + \sin 12)$

(f) $\frac{1}{17}e^{-3}(4\cos 12 + \sin 12)$

(g) The integral does not converge.

(h) None of the above

Answer: $\frac{1}{17}e^{-3}(4\cos 12 + \sin 12)$

7. Find the power series expansion for the function $\sin^{-1} x$ (or $\arcsin x$) expanded about 0.

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$

(b) $\displaystyle\sum_{k=0}^{\infty} \binom{-1/2}{k} \frac{(-1)^k}{2k+1} x^{2k+1}$

(c) $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}$

(d) $\displaystyle\sum_{k=0}^{\infty} \binom{1/2}{k} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

(e) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} x^{k+1}$

(f) $\displaystyle\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$

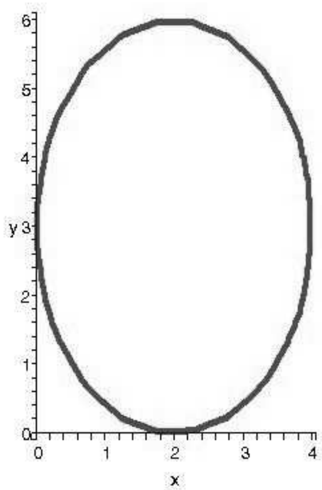
(g) $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}$

(h) $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

(i) None of the above

Answer: $\sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} (-1)^k \frac{x^{2k+1}}{2k+1}$.

8. Identify the equation that best goes with the following graph in rectangular coordinates.



(a) $\frac{(x-3)^2}{9} + \frac{(y-2)^2}{4} = 1$

(b) $\frac{(y-3)^2}{9} - \frac{(x-2)^2}{4} = 1$

(c) $x = y^2 - 2$

(d) $\frac{(x-2)^2}{4} + \frac{(y-3)^2}{9} = 1$

(e) $y = 4 - x^2$

(f) $\frac{(x-3)^2}{9} - \frac{(y-2)^2}{4} = 1$

(g) $\frac{(x-2)^2}{4} - \frac{(y-3)^2}{9} = 1$

(h) $\frac{(y-2)^2}{4} - \frac{(x-3)^2}{9} = 1$

(i) None of the above

Answer: $\frac{(x-2)^2}{4} + \frac{(y-3)^2}{9} = 1$

9. Which of the following integrals represents the surface area of the surface generated by revolving the curve $y = e^{2x}$, $0 \leq x \leq 1$, about the line $y = -2$.

(a) $\int_0^1 2\pi(e^{2x} - 2)\sqrt{1 + 4e^{4x}}\, dx$

(b) $\int_0^1 2\pi(e^{2x} + 2)\sqrt{1 + 4e^{4x}}\, dx$

(c) $\int_0^1 2\pi(e^{2x})\sqrt{1 + 4e^{4x}}\, dx$

(d) $\int_0^1 2\pi(e^{2x})\sqrt{1 + e^{4x}}\, dx$

(e) $\int_0^1 2\pi(e^{2x} - 2)\sqrt{1 + e^{4x}}\, dx$

(f) $\int_0^1 2\pi(e^{2x} - 2)\, dx$

(g) $\int_0^1 2\pi(e^{2x} + 2)\, dx$

(h) $\int_0^1 2\pi(e^{2x})\, dx$

(i) None of the above

Answer: $\int_0^1 2\pi(e^{2x} + 2)\sqrt{1 + 4e^{4x}}\, dx$

The answers to the multiple choice MUST be entered on the grid on the previous page. Otherwise, you will not receive credit.

Part II: Written Solutions

For problems 10 – 18, write your answers in the space provided. Neatly show your work for full credit.

10. Find a formula for $\int \sqrt{b^2 - a^2x^2}\, dx$. Here a, b are positive constants.

Let $ax = b\sin t$, then $a\,dx = b\cos t\,dt$, so

$$\begin{aligned}
\int \sqrt{b^2 - a^2x^2}\, dx &= \int \sqrt{b^2 - b^2\sin^2 t}\,\frac{b}{a}\cos t\,dt \\
&= \frac{b^2}{a}\int \cos^2 t\,dt \\
&= \frac{b^2}{a}\int \frac{1}{2}(1+\cos 2t)\,dt \\
&= \frac{b^2}{2a}\left(t + \frac{1}{2}\sin 2t\right) + C \\
&= \frac{b^2}{2a}\left(\sin^{-1}\frac{ax}{b} + \frac{ax}{b}\sqrt{1 - \left(\frac{ax}{b}\right)^2}\right) + C \\
&= \frac{b^2}{2a}\sin^{-1}\frac{ax}{b} + \frac{x}{2}\sqrt{b^2 - a^2x^2} + C
\end{aligned}$$

11. Find the area of the region bounded by the curve $x = y - y^2$ and the line $y = -x$.

The curve and the line intersect when $x = -x - (-x)^2$ or at $x = 0, y = 0$ and $x = -2, y = 2$. Area of region A is given by

$$A = \int_0^2 (y - y^2) - y\,dy = y^2 - \frac{1}{3}y^3\Big|_0^2 = \frac{4}{3}.$$

12. Find the volume of the solid generated by revolving the region enclosed by $y = 4$ and $y = 3(x-3)^2 + 1$ about the y-axis.

The two curves intersect at $(2, 4)$ and $(4, 4)$. Using the shell method, volume V is given by

$$V = \int_2^4 2\pi x(4 - (3(x-3)^2 + 1))\,dx = 2\pi\int_2^4 (-3x)(8 - 6x + x^2)\,dx = -6\pi\left[4\,x^2 + \frac{1}{4}\,x^4 - 2\,x^3\right]_2^4$$

so

$$V = 24\pi$$

13. Determine the values of p for which the integral $\displaystyle\int_2^\infty \frac{1}{x\,(\ln x)^p}\,dx$ converges. Justify your answer.

Let $u = \ln x$, then $du = \dfrac{1}{x}\,dx$ and so the integral may be written as

$$\int_2^\infty \frac{1}{x(\ln x)^p}\,dx = \int_{\ln 2}^\infty \frac{1}{u^p}\,du$$

The above improper integral is convergent if $p > 1$, thus the integral $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^p}\,dx$ converges for $p > 1$.

14. (a) Find a Maclaurin series which represents the function $\dfrac{\sin\sqrt{x}}{\sqrt{x}}$ when $x > 0$.

(b) Hence calculate $\displaystyle\lim_{x\to 0+} \frac{\sin\sqrt{x}}{\sqrt{x}}$.

(c) Find the interval of convergence of this power series.

From the Maclaurin series for the sine function,

$$\sin t = t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \cdots = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}t^{2n+1}$$

$$\begin{aligned}
\frac{\sin\sqrt{x}}{\sqrt{x}} &= \frac{1}{\sqrt{x}}\left(\sqrt{x} - \frac{1}{3!}(\sqrt{x})^3 + \frac{1}{5!}(\sqrt{x})^5 + \cdots\right) \\
&= 1 - \frac{1}{3!}(\sqrt{x})^2 + \frac{1}{5!}(\sqrt{x})^4 + \cdots \\
&= 1 - \frac{1}{3!}x + \frac{1}{5!}x^2 + \cdots \\
&= \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}x^n
\end{aligned}$$

Hence

$$\lim_{x\to 0+} \frac{\sin\sqrt{x}}{\sqrt{x}} = \lim_{x\to 0+} 1 - \frac{1}{3!}x + \frac{1}{5!}x^2 + \cdots = 1$$

Interval of convergence: from ratio test, the series is (absolutely) convergent if

$$\left|\frac{\frac{(-1)^{n+1}}{(2(n+1)+1)!}x^{n+1}}{\frac{(-1)^n}{(2n+1)!}x^n}\right| < 1$$

as $n \to \infty$. So

$$\left|\frac{x}{(2n+3)(2n+2)}\right| < 1 \Rightarrow |x| < (2n+3)(2n+2) \to \infty$$

so the radius of convergence is infinite.

15. Find the Taylor polynomial of degree 3 for $f(x) = 4x^3 + 3x^2 + 2x + 1$ which is centered at 1.

Applying Taylor series expansion at $x = 1$, and noting that the series terminates after the cubic term,

$$f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

Now

$$\begin{array}{l|l} f(x) = 4x^3 + 3x^2 + 2x + 1 & f(1) = 10 \\ f'(x) = 12x^2 + 6x + 2 & f'(1) = 20 \\ f''(x) = 24x + 6 & f''(1) = 30 \\ f'''(x) = 24 & f'''(1) = 24 \end{array}$$

so

$$\begin{aligned} f(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 \\ &= 10 + 20(x-1) + \frac{30}{2!}(x-1)^2 + \frac{24}{3!}(x-1)^3 \\ &= 10 + 20\ (x-1) + 15\ (x-1)^2 + 4\ (x-1)^3 \end{aligned}$$

16. Compute $I_1 = \int x \ln x \, dx$ and determine a reduction formula for

$$I_n = \int x(\ln x)^n \, dx, \quad n > 1$$

Integrating by parts, with $u = \ln x,\ dv = x\,dx$,

$$I_1 = (\ln x)\frac{x^2}{2} - \int \frac{x^2}{2}\frac{1}{x}\,dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$$

For $n > 1$, with $u = (\ln x)^n,\ dv = x\,dx$,

$$I_n = (\ln x)^n \frac{x^2}{2} - \int \frac{x^2}{2} n(\ln x)^{n-1}\frac{1}{x}\,dx = \frac{1}{2}x^2(\ln x)^n - \frac{n}{2}\int x(\ln x)^{n-1}\,dx$$

and thus

$$I_n = \frac{1}{2}x^2(\ln x)^n - \frac{n}{2}I_{n-1}.$$

17. Sketch the closed curve $r = 7\cos\left(\theta - \frac{\pi}{4}\right)$ and determine the area enclosed by the curve.

The equation represents a circle with radius= $7/2$, passing the origin and center along $y = x$.

18. (a) For which values of x does $\sum_{k=1}^{\infty} \frac{1}{k}(1-e^x)^k$ converge?

(b) What is the sum of this series?

The series obviously converges when $x = 0$.

Consider the series $S(t) = \sum_{k=1}^{\infty} \frac{t^k}{k}$. Now

$$S'(t) = \sum_{k=1}^{\infty} t^{k-1} = \sum_{k=0}^{\infty} t^k$$

For $|t| < 1$, $\quad S'(t) = \dfrac{1}{1-t}$ and so the series converges.

Hence the series $\sum_{k=1}^{\infty} \frac{1}{k}(1-e^x)^k$ converges when $x = \ln 2$ or when

$$|1-e^x| < 1 \Rightarrow -1 < e^x - 1 < 1 \Rightarrow x < \ln 2.$$

Also for $|t| < 1$,

$$S'(t) = \frac{1}{1-t} \Rightarrow S(t) = \int \frac{dt}{1-t} = -\ln(1-t)$$

since $S(0) = 0$. Consequently,

$$\sum_{k=1}^{\infty} \frac{1}{k}(1-e^x)^k = S(1-e^x) = -\ln(e^x) = -x.$$