Name ___________KEY___________ Instructor________________________Section No. ________

Student Number ________________________

Math 113/113H – Winter 2006

Departmental Final Exam

Instructions:

- The time limit is 3 hours.
- Problems 1-6 short-answer questions, each worth 2 points.
- Problems 7 through 13 are multiple choice questions, each worth 3 points.
- For problems 14 through 24, give the best answer and *justify* it by giving suitable reasons and/or by showing relevant work.
- Work on scratch paper will not be graded.
- Please write neatly.
- Notes, books, and calculators are not allowed.
- Expressions such as $\ln(1)$, e^0, $\sin(\pi/2)$, etc. must be simplified for full credit.

For administrative use only:

1–6	/12
M.C.	/21
14	/4
15	/4
16	/4
17	/6
18	/4
19	/8
20	/4
21	/8
22	/8
23	/9
24	/8
Total	/100

Short Answer.

1. $\int \ln x \, dx =$ $\underline{\qquad x \ln x - x + C \qquad}$

2. $\int_0^{\pi/2} \sin x \, dx =$ $\underline{\qquad -\cos x|_0^{\pi/2} = 0 - (-1) = 1 \qquad}$

3. $\int \tan x \sec^2 x \, dx =$ $\underline{\qquad \frac{1}{2}\tan^2 x + C \text{ or } \frac{1}{2}\sec^2 x + C \qquad}$

4. The Taylor series for e^{2x}, centered at 0, is $\underline{1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}}$

5. $\displaystyle\int \frac{x^2}{\sqrt{x^2+4}} \, dx$ can be integrated using the trigonometric substitution $x =$ $\underline{\ 2\tan\theta\ }$

6. The definite integral $\int_0^3 2\pi x\sqrt{x^2+1} \, dx$ represents the volume of the solid of revolution generated by revolving the curve $y =$ $\underline{\qquad \sqrt{x^2+1} \qquad}$, $x \in [\ \underline{\ 0\ }\ ,\ \underline{\ 3\ }\ ]$, about the y-axis.

Multiple Choice. Use the following chart to show your choices for Problems 7–13. Shade in your choice. Only this chart will be graded on these problems.

Problem 7	(a)	(b)	■	(d)	(e)	(f)	(g)	(h)	(i)	(j)
Problem 8	(a)	(b)	■	(d)	(e)	(f)	(g)	(h)	(i)	(j)
Problem 9	(a)	(b)	(c)	(d)	■	(f)	(g)	(h)	(i)	(j)
Problem 10	(a)	(b)	(c)	(d)	■	(f)	(g)	(h)	(i)	(j)
Problem 11	■	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)
Problem 12	(a)	(b)	■	(d)	(e)	(f)	(g)	(h)	(i)	(j)
Problem 13	(a)	(b)	(c)	■	(e)	(f)	(g)	(h)	(i)	(j)

7. $\displaystyle\int_1^{\infty} \frac{1}{x^p} \, dx$

(a) converges if $0 < p < 1$ (b) converges if $p = 1$ (c) converges if $p > 1$

(d) diverges if $p > 1$ (e) diverges if $p > 0$ (f) diverges if $p \neq 1$

(g) none of these

8. The area of the surface of revolution generated by revolving the curve $y = \sin x, 0 \leq x \leq \pi$, about the line $y = -1$ is

(a) $\displaystyle\int_0^\pi 2\pi \sin x \sqrt{1+\cos^2 x}\, dx$ (b) $\displaystyle\int_0^\pi 2\pi \sin x \sqrt{1+\sin^2 x}\, dx$

(c) $\displaystyle\int_0^\pi 2\pi(1+\sin x)\sqrt{1+\cos^2 x}\, dx$ (d) $\displaystyle\int_0^\pi 2\pi(1-\sin x)\sqrt{1+\sin^2 x}\, dx$

(e) $\displaystyle\int_0^\pi 2\pi(\sin x - 1)\sqrt{1+\sin^2 x}\, dx$ (f) $\displaystyle\int_0^{2\pi} \pi \sin x \sqrt{1+\cos^2 x}\, dx$

(g) $\displaystyle\int_0^{2\pi} \pi(1+\sin x)\sqrt{1+\sin^2 x}\, dx$ (h) none of these

9. The value of $\displaystyle\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx$ is

(a) $\frac{1}{2}a$ (b) $\frac{1}{4}a^2$ (c) a (d) πa^2 (e) $\frac{1}{2}\pi a^2$ (f) $\frac{1}{4}\pi a^2$

(g) $\frac{1}{2}\pi^2$ (h) $\frac{1}{4}\pi^2 a$ (i) $-\frac{1}{2}\pi a^2$ (j) none of these

10. Region R lies in the first quadrant, has area 6, and has centroid (4, 7). What is the volume of the solid generated by revolving R about the line $x = -1$?

(a) 36π (b) 42π (c) 48π (d) 54π (e) 60π (f) 72π (g) 84π

(h) 96π (i) none of these

11. Which best describes how the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$ behaves?

(a) converges absolutely (b) converges conditionally, not absolutely (c) doesn't converge

(d) converges to infinity (e) converges to negative infinity

Answers to Problems 8–11 must be shaded in the chart on Page 1 to be counted.

12. By using two non-zero terms of the Taylor series, the integral $\int_0^1 \sin(x^2)\ dx$ is approximately (to two digit accuracy)

(a) .96 (b) .43 (c) .31 (d) .29 (e) .12 (f) .03

13. $\int e^x \cos x\ dx =$

(a) $e^x \sin x + C$ (b) $e^x \cos x - e^x \sin x + C$ (c) $e^x \sin x - e^x \sin 2x + C$

(d) $\frac{1}{2}e^x \sin x + \frac{1}{2}e^x \cos x + C$ (e) $2e^x \sin x - 2e^x \cos x + C$ (f) none of the above

Answers to Problems 12 and 13 must be shaded in the chart on Page 1 to be counted.

Essay Problems Work Problems 14–24 as you would homework problems, showing your steps and justifying them.

14. Find a formula for $\displaystyle\int \sqrt{1-a^2x^2}\ dx$ $(a > 0)$. $x = \frac{1}{a}\sin\theta, dx = \frac{1}{a}\cos\theta\ d\theta, \sqrt{1-a^2x^2} = \cos\theta \Longrightarrow$

$$\begin{aligned}
\int \sqrt{1-a^2x^2}\ dx &= \int \cos\theta \cdot \frac{1}{a}\cos\theta\ d\theta \\
&= \frac{1}{a}\int \cos^2\theta\ d\theta \\
&= \frac{1}{2a}\int (1+\cos 2\theta)\ d\theta \\
&= \frac{1}{2a}[\theta + \frac{1}{2}\sin 2\theta] + C \\
&= \frac{1}{2a}[\theta + \sin\theta\cos\theta] + C \\
&= \frac{1}{2a}[\sin^{-1}(ax) + ax\sqrt{1-a^2x^2}] + C
\end{aligned}$$

15. Find the length of the curve $y = \ln(\sin x), x \in [\frac{1}{6}\pi, \frac{1}{4}\pi]$. $y' = \frac{\cos x}{\sin x} = \cot x$;

$$\begin{aligned}
s = \int_{\pi/6}^{\pi/4} \sqrt{1+\cot^2 x}\ dx &= \int_{\pi/6}^{\pi/4} \csc x\ dx \\
&= \ln|\csc x - \cot x|\Big|_{\pi/6}^{\pi/4} \\
&= \ln(\sqrt{2}-1) - \ln(2-\sqrt{3})
\end{aligned}$$

16. Evaluate $\displaystyle\int_0^{\pi/2} \sin^3 x \cos^2 x \, dx$.

$$\begin{aligned}
\int_0^{\pi/2} \sin^3 x \cos^2 x \, dx &= \int_0^{\pi/2} \sin^2 x \cos^2 x \sin x \, dx \\
&= \int_0^{\pi/2} (1-\cos^2 x)\cos^2 x \sin x \, dx \\
&= -\int_1^0 (1-u^2)u^2 du \qquad [u = \cos x, du = -\sin x \, dx] \\
&= \int_0^1 (u^2 - u^4) du \\
&= \frac{1}{3}u^3 - \frac{1}{5}u^5 \Big|_0^1 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}
\end{aligned}$$

17. A swimming pool has a circular window of radius 1.2 meters in a side wall. When the water in the pool exactly covers the lower half of the window, what is the force of the water pressure on the window? (Assume that the side wall is vertical; give your answer in terms of the weight-density w of the water.)

Let $r = 1.2$ be the radius of the window. A horizontal strip a distance h below the diameter of the window has length $l = 2\sqrt{r^2 - h^2}$. Therefore the force on the window is

$$\begin{aligned}
F &= \int_0^r wh \cdot 2\sqrt{r^2 - h^2} \, dh \\
&= -w \int_{r^2}^0 u^{1/2} du \qquad [u = r^2 - h^2, du = -2h dh] \\
&= \frac{2}{3} w u^{3/2} \Big|_0^{r^2} \\
&= \frac{2}{3} w r^3 = 1.152 w
\end{aligned}$$

18. Find $\displaystyle\int \frac{x+1}{x^2-4} \, dx$.

$\dfrac{x+1}{x^2-4} = \dfrac{A}{x+2} + \dfrac{B}{x-2} \Longrightarrow x+1 = A(x-2) + B(x+2) = (A+B)x + 2(B-A) \Longrightarrow A = \frac{1}{4}, B = \frac{3}{4}$.
Therefore

$$\int \frac{x+1}{x^2-4} \, dx = \int \left(\frac{1/4}{x+2} + \frac{3/4}{x-2} \right) dx = \frac{1}{4} \ln|x+2| + \frac{3}{4} \ln|x-2| + C$$

19. Determine whether each integral converges, and give its value if it does so.

(a) $\displaystyle\int_0^\infty x^2 e^{-x^3}\,dx$

$$\begin{aligned}
\int_0^\infty x^2 e^{-x^3}\,dx &= \lim_{b\to\infty}\int_0^b x^2 e^{-x^3}dx \\
&= \lim_{b\to\infty} -\frac{1}{3}\int_0^{-b^3} e^w dw \qquad [w=-x^3, dw=-3x^2dx] \\
&= \lim_{b\to\infty}\frac{1}{3}(1-e^{-b^3}) = \frac{1}{3}
\end{aligned}$$

Therefore the integral converges to $\frac{1}{3}$.

(b) $\displaystyle\int_{-1}^1 \frac{1}{\sqrt{|x|}}\,dx$

By symmetry,

$$\begin{aligned}
\int_{-1}^1 \frac{1}{\sqrt{|x|}}\,dx &= 2\int_0^1 x^{-1/2}dx \\
&= \lim_{a\to 0+}\int_a^1 2x^{-1/2}dx \\
&= \lim_{a\to 0+} 4\sqrt{x}\big|_a^1 \\
&= \lim_{a\to 0+}(4-4\sqrt{a}) = 4
\end{aligned}$$

Therefore the integral converges to 4.

20. Find the Taylor series for the function $f(x)=x^2+2x+5$ centered at the point $x=2$. Show your work.

$f(x) = x^2+2x+5, f'(x) = 2x+2, f''(x) = 2, f'''(x) = 0 \implies f(2) = 13, f'(2) = 6, f''(2) = 2, f'''(2) = 0$. Hence

$$f(x) = 13+6(x-2)+\frac{2}{2!}(x-2)^2 = 13+6(x-2)+(x-2)^2.$$

[Note that there are different ways to do this.]

21. For each of the following power series, determine what function it converges to, and find the interval of convergence (watch the end-points).

(a) $\displaystyle\sum_{n=1}^{\infty} nx^{n-1}$

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}.$$

The geometric series converges on $(-1,1)$. At both –1 and 1, we get divergent series, so $(-1,1)$ is the interval of convergence.

(b) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{n!}$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{n!} = \sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!} = e^{-x^3}.$$

This series converges for all $x \in \mathbb{R}$.

22. (a) Give the formal definition of the statement $\displaystyle\lim_{n\to\infty} a_n = L$.

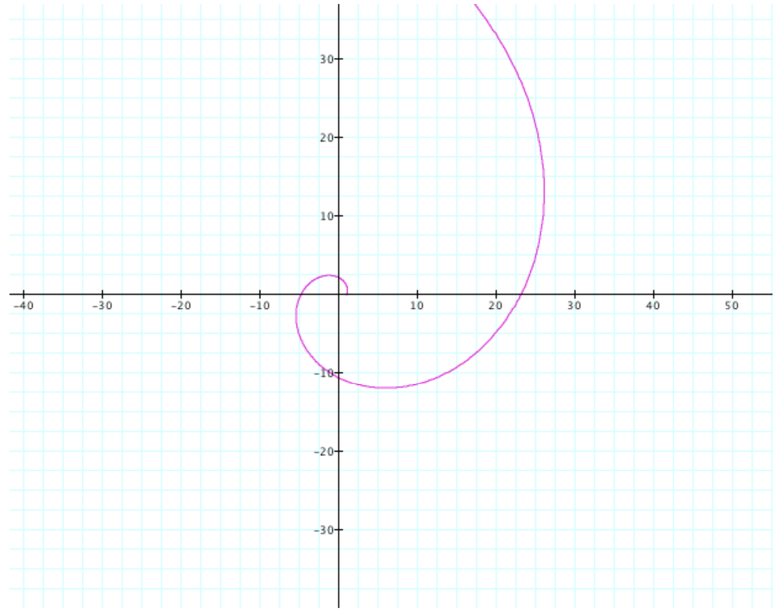
$$\lim_{n\to\infty} a_n = L \iff \forall \epsilon > 0, \exists k \in \mathbb{N} : n > k \implies |a_n - L| < \epsilon.$$

(b) Determine the limit $\displaystyle\lim_{n\to\infty}\left(1+\frac{\pi}{n}\right)^n$ (by any means).

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{\pi}{n}\right)^n &= \exp\left(\lim_{n\to\infty} n\ln(1+\frac{\pi}{n})\right) = \exp\left(\lim_{n\to\infty}\frac{\ln(1+\frac{\pi}{n})}{1/n}\right)\\ &= \exp\left(\lim_{n\to\infty}\frac{-(\pi/n^2)/\ln(1+\pi/n)}{-1/n^2}\right) = \exp\left(\pi\lim_{n\to\infty}\frac{1}{1+\pi/n}\right) = e^{\pi}\end{aligned}$$

Or use Theorem 117(5).

23. (a) Sketch the graph of $r = e^{\theta/2}$.



(b) Find the area inside the curve $r = e^{\theta/2}$ and outside the circle $r = 1$ for $0 \le \theta \le \pi$.

$A = \int_0^\pi \frac{1}{2} r^2 d\theta - \frac{1}{2}\pi = \int_0^\pi \frac{1}{2} e^\theta d\theta - \frac{1}{2}\pi = \frac{1}{2} e^\theta|_0^\pi - \frac{1}{2}\pi = \frac{1}{2}e^\pi - \frac{1}{2} - \frac{1}{2}\pi.$

(c) Find the slope of the polar curve $r = e^{\theta/2}$ at the point [1, 0].

$$y = e^{\theta/2}\sin\theta \Longrightarrow \frac{dy}{d\theta} = \frac{1}{2}e^{\theta/2}\sin\theta + e^{\theta/2}\cos\theta \Longrightarrow \frac{dy}{d\theta}|_{\theta=0} = 1;$$

$$x = e^{\theta/2}\cos\theta \Longrightarrow \frac{dx}{d\theta} = \frac{1}{2}e^{\theta/2}\cos\theta - e^{\theta/2}\sin\theta \Longrightarrow \frac{dx}{d\theta}|_{\theta=0} = \frac{1}{2};$$

$$\frac{dy}{dx} = \frac{1}{1/2} = 2.$$

24. The position, in feet, of a slow pitch softball at time t, in seconds, is given by the parametric equations

$$\begin{cases} x = 18\sqrt{3}\,t \\ y = -16t^2 + 18t + 4 \end{cases}.$$

(a) What is the rate of change of the height of the ball with respect to its horizontal position when it crosses the plate at $t = 1.2$ seconds?

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-32t+18}{18\sqrt{3}};\ \frac{dy}{dx}\bigg|_{t=1.2} = -\frac{20.4}{18\sqrt{3}}$$

(b) Set up but do not evaluate an integral giving the arc length of the path of the ball for $0 \le t \le 1.2$.

$$s = \int_0^{1.2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{1.2} \sqrt{18^2(3) + (18-32t)^2}\,dt$$