# Math 113 – Winter 2007 — Key

Departmental Final Exam

PART I: SHORT ANSWER AND MULTIPLE CHOICE QUESTIONS

Do not show your work for problems in this part.

1. Fill in the blanks with the correct answer.

   (a) $1 + x + x^2 + \cdots$ is the Maclaurin series for $\boxed{\dfrac{1}{1-x}}$

   (b) $\displaystyle\int \ln(x)\,dx =$ $\boxed{x \ln x - x + C}$

   (c) What technique must be used to integrate $\displaystyle\int \frac{x+1}{x^3+4x}dx$?

   $\boxed{\text{partial fraction}}$

   (d) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$ is the Maclaurin series for $\boxed{\sin x}$

   (e) What substitution should be used to find $\displaystyle\int \sqrt{4-x^2}dx$ $\boxed{x = 2\sin t \quad \text{or} \quad x = 2\cos t}$

2. True/False: Write T if statement always holds, F otherwise.
   Let $\sum a_n = \sum_{n=1}^{\infty} a_n$ be an arbitrary series.

   (a) $\boxed{\text{F}}$ $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges absolutely.

   (b) $\boxed{\text{T}}$ $\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$

   (c) $\boxed{\text{T}}$ The improper integral $\int_1^{\infty} \frac{dx}{x^3+1}$ converges.

   (d) $\boxed{\text{T}}$ If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then it converges.

   (e) $\boxed{\text{F}}$ If $\sum_{n=1}^{\infty} a_n^2$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

*Problems 3 through 7 are multiple choice. Each multiple choice problem is worth 3 points. In the grid below fill in the square corresponding to each correct answer.*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 3 | A | B | C | D | ■ | F | G | H | I |
| 4 | A | B | C | ■ | E | F | G | H | I |
| 5 | A | ■ | C | D | E | F | G | H | I |
| 6 | A | B | C | ■ | E | F | G | H | I |
| 7 | ■ | B | C | D | E | F | G | H | I |

3. Find the length of the graph of $y = x^{1/2} - \frac{x^{3/2}}{3}$ for $x \in [0, 1]$.

(a) $\frac{4}{3}\sqrt{2}$ (b) $\frac{2}{3}$ (c) $\frac{8}{3}$

(d) $\frac{8}{9}\sqrt{3}$ (e) $\boxed{\frac{4}{3}}$ (f) None of the above.

4. Find the area between $y = x$ and $y = x^2$ for $x \in [-1, 3]$.

(a) $\frac{29}{6}$ (b) $-\frac{9}{2}$ (c) $\frac{9}{2}$

(d) $\boxed{\frac{17}{3}}$ (e) $\frac{41}{3}$ (f) None of the above.

5. Let $A$ denote the region between the graph of $y = \cos x$ and the $x$ axis for $x \in \left[0, \frac{\pi}{2}\right]$. A solid is obtained by revolving $A$ about the line $x = -2$. Find the volume of this solid.

(a) $2\pi\left(\frac{1}{2}\pi - 3\right)$ (b) $\boxed{2\pi\left(\frac{1}{2}\pi + 1\right)}$ (c) $2\pi\left(\frac{1}{2}\pi - 1\right)$

(d) $4\pi$ (e) $2\pi\left(\frac{1}{2}\pi + 2\right)$ (f) None of the above.

The method of shells is the way to do this fascinating and glorious exercise. You need to evaluate $2\pi\int_0^{\pi/2}(x+2)\cos(x)\,dx = 2\pi\left(\frac{1}{2}\pi + 1\right) = 2\pi + \pi^2$.

6. $\displaystyle\int_0^2 \frac{x^2}{\sqrt{4-x^2}}dx =$

(a) $3\pi/2$ (b) $\pi/3$ (c) $\pi/2$

(d) $\boxed{\pi}$ (e) $2\pi$ (f) None of the above.

7. Find the area enclosed by the spiral $r = f(\theta) = e^{\theta}$ where $\theta \in [0, \pi]$.

(a) $\boxed{\frac{1}{4}e^{2\pi} - \frac{1}{4}}$ (b) $\frac{1}{2}e^{2\pi} - \frac{1}{2}$ (c) $e^{\pi} - 1$

(d) $\frac{1}{2}e^{\pi} - \frac{1}{2}$ (e) $\frac{1}{2}e^{2\pi} - 1$ (f) None of the above.

$$\frac{1}{2}\int_0^{\pi} e^{2\theta}\, d\theta = \frac{1}{4}e^{2\pi} - \frac{1}{4}$$

*The answers to the multiple choice MUST be entered on the grid on the previous page. Otherwise, you will not receive credit.*

## Part II: Written Solutions

*For problems 8 – 18, write your answers in the space provided. Neatly show your work for full credit.*

8. Determine whether each series converges absolutely, conditionally or fails to converge. State your conclusion next to the series.

(a) $\displaystyle\sum_{n=11}^{\infty} \frac{1}{(n-10)\ln n}$

$$\sum_{n=11}^{\infty} \frac{1}{(n-10)\ln n} = \sum_{m=1}^{\infty} \frac{1}{(m)\ln(m+10)}$$

Using integral test, we compare the series with the integral $\displaystyle\int_1^{\infty} \frac{1}{x\ln(x+10)}\,dx$. With substitution $u = \ln(x+10)$,

$$\int_1^{\infty} \frac{1}{x\ln(x+10)}\,dx = \lim_{M\to\infty} \int_1^{M} \frac{1}{x\ln(x+10)}\,dx = \lim_{M\to\infty} \ln(\ln(M) - \ln(\ln 11) = \infty$$

So the series diverges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2\sin(n)}{n^2+1}$

$$\sum_{n=1}^{\infty} \left|\frac{2\sin(n)}{n^2+1}\right| \le \sum_{n=1}^{\infty} \frac{2}{n^2}$$

So series converges absolutely.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{n\cos(n\pi)}{n^2+1}$

$$\sum_{n=0}^{\infty} \frac{n\cos(n\pi)}{n^2+1} = \sum_{m=0}^{\infty} \frac{2m(-1)^m}{4m^2+1}$$

Alternating series with $\dfrac{2m}{4m^2+1} \to 0$ as $m \to \infty$, so series converges conditionally.

9. Use a Maclaurin series to estimate

$$\int_0^{0.06} \frac{(e^x - 1)}{x} dx$$

to four digit accuracy.

(Simple application of series)

$\frac{(e^x-1)}{x} = 1 + \frac{1}{2}x + \frac{1}{6}x^2 + O(x^3)$ and so an approximation is

$\int_0^{.06} \left(1 + \frac{1}{2}x + \frac{1}{6}x^2\right) dx = .0\,609\,12$

10. Give the Maclaurin series for

$$G(x) = \frac{1}{\sqrt{x^2+1}}$$

and determine the radius of convergence.

This is a binomial series problem.

$(1+x^2)^{-1/2} = \sum_{k=0}^{\infty} \binom{-1/2}{k} x^{2k}$ and the radius of convergence is 1.

11. Find the power series of

$$f(x) = \frac{1}{x+3}$$

which is centered at 1 and give its interval of convergence.

This is a geometric series expanded about a point not equal to 0

$\frac{1}{x+3} = \frac{1}{x-1+4} = \frac{1}{4}\frac{1}{1+\frac{x-1}{4}} = \frac{1}{4}\sum_{k=0}^{\infty} (-1)^k \left(\frac{x-1}{4}\right)^k.$

12. Find $\int x^2 e^{-x} dx$ .

Using integration by parts,

$$\int x^2 e^{-x} dx = -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

13. Evaluate the improper integral $\int_1^\infty \frac{dx}{x^2(x+1)}$ or show that it does not converge.

Using partial fractions and improper integrals:

$$\begin{aligned}
\int_1^\infty \frac{dx}{x^2(x+1)} &= \lim_{M\to\infty} \int_1^M \left( \frac{1}{x+1} + \frac{1}{x^2} - \frac{1}{x} \right) dx \\
&= \lim_{M\to\infty} \ln(x+1) - \frac{1}{x} - \ln x \Big|_1^M \\
&= \lim_{M\to\infty} \ln\left(\frac{M+1}{M}\right) - \frac{1}{M} - \ln 2 + 1 \\
&= 1 - \ln 2
\end{aligned}$$

14. Use partial fraction decomposition to evaluate the following integral:

$$\int \frac{6x^2 - 3x - 18}{(x-5)(x^2+2x+4)}\,dx$$

$$\frac{6x^2 - 3x - 18}{(x-5)(x^2+2x+4)} = \frac{3}{x-5} + 3\frac{2+x}{x^2+2x+4}$$

and so

$$\begin{aligned} &\int \frac{6x^2 - 3x - 18}{(x-5)(x^2+2x+4)}\,dx \\ =\ & 3\ln|x-5| + \frac{3}{2}\ln\left|x^2+2x+4\right| + \sqrt{3}\arctan\frac{1}{6}(2x+2)\sqrt{3} + C \end{aligned}$$

15. (a) Find the sum of the series $S = \sum_{n=1}^{\infty} \frac{1}{2^{2n-1}}$.

The series is a geometric series $\sum a\, r^n$

$$\sum_{n=1}^{\infty} \frac{1}{2^{2n-1}} = 2\sum_{n=1}^{\infty} \frac{1}{4^n} = 2\frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{2}{3}$$

(b) Determine the convergence of the series $S = \sum_{n=1}^{\infty} \frac{1}{(2n-1)2^{2n-1}}$.

Converges by the ratio test:

$$a_n = \frac{1}{(2n-1)2^{2n-1}}$$

$$\frac{a_{n+1}}{a_n} = \frac{(2n-1)2^{2n-1}}{(2n+1)2^{2n+1}} = \frac{1}{4}\frac{(2n-1)}{(2n+1)}$$

As $n \to \infty$,

$$\frac{a_{n+1}}{a_n} \to \frac{1}{4} < 1.$$

Or, more simply, by comparison test using result in part (a):
For $n > 1$,

$$\frac{1}{(2n-1)2^{2n-1}} < \frac{1}{2^{2n-1}}$$

16. Find the interval of convergence of the power series $S = \sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$.

For convergence,

$$\left| \frac{\frac{(x-3)^{n+1}}{n+1}}{\frac{(x-3)^n}{n}} \right| < 1 \Rightarrow \quad |x-3| < \frac{n+1}{n}$$

As $n \to \infty$, $\frac{n+1}{n} \to 1$, so

$$|x-3| < 1 \Rightarrow x \in (2,4)$$

For $x = 4$, we have the divergent series $\sum \frac{1}{n}$.

For $x = 2$, we have the convergent alternating series $\sum \frac{(-1)^n}{n}$.

So the answer is $[2,4)$.

17. Find the power series of $\ln(1-x)$ centered at 0 and give its radius of convergence.

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 + O\left(x^6\right) = -\sum_{k=1}^{\infty} \frac{x^k}{k}.$$

Radius of convergence: 1

18. Consider the ellipse, $x^2 - 6x + 2y^2 + 16y + 37 = 0$. Find the vertices, express in standard form, and sketch the graph.

By completing the square,
$(x-3)^2 + 2(y+4)^2 - 9 - 32 + 37 = 0.$
$(x-3)^2 + 2(y+4)^2 = 4$ and so $\frac{(x-3)^2}{4} + \frac{(y+4)^2}{2} = 1.$

*—End—*