Math 113 (Calculus II)
Final Exam Form A KEY

Multiple Choice. Fill in the answer to each problem on your scantron. Make sure your name, section and instructor is on your scantron.

1. Find the area of the region enclosed by $y = x$ and $y = 5x - x^2$.

   a) $\frac{5}{3}$ b) $\frac{8}{3}$ c) $\frac{16}{3}$

   d) $\frac{28}{3}$ e) $\frac{32}{3}$ f) $\frac{80}{3}$

   Solution: e)

2. Set up the integral representing the volume of the solid obtained by rotating the region bounded by $y = x^2 + 1$ and $y = 3 - x^2$ about the x-axis.

   a) $\displaystyle\int_{-1}^{1} \pi[(3 - x^2)^2 - (x^2 + 1)^2]\, dx$ b) $\displaystyle\int_{-1}^{1} 2\pi x\,[(3 - x^2) - (x^2 + 1)]\, dx$

   c) $\displaystyle\int_{-1}^{1} \pi[(x^2 + 1)^2 - (3 - x^2)^2]\, dx$ d) $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} 2\pi x\,[(x^2 + 1) - (3 - x^2)]\, dx$

   e) $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \pi[(x^2 + 1)^2 - (3 - x^2)^2]\, dx$ f) none of the above

   Solution: a)

3. Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5 x \cos^3 x\, dx$.

   a) $\frac{1}{24}$ b) $\frac{1}{6}$ c) $\frac{1}{8}$

   d) $-\frac{1}{8}$ e) $-\frac{1}{6}$ f) $-\frac{1}{24}$

   Solution: a)

4. Determine whether $\displaystyle\int_0^{\infty} \frac{1}{1 + x^2}\, dx$ is convergent or divergent. If convergent, evaluate the integral.

   a) divergent b) 0, convergent c) $\frac{\pi}{4}$, convergent

   d) $\frac{\pi}{2}$, convergent e) π, convergent f) 2π, convergent

   Solution: d)

5. Set up, but do not evaluate, an integral for the area of the surface obtained by rotating the curve $y = e^{2x}$, $0 \le x \le 1$ about the x-axis.

a) $\displaystyle\int_0^1 2\pi x\sqrt{1+e^{4x}}\,dx$ b) $\displaystyle\int_0^1 2\pi x\sqrt{1+2e^{2x}}\,dx$ c) $\displaystyle\int_0^1 2\pi x\sqrt{1+4e^{4x}}\,dx$

d) $\displaystyle\int_0^1 2\pi e^{2x}\sqrt{1+e^{4x}}\,dx$ e) $\displaystyle\int_0^1 2\pi e^{2x}\sqrt{1+2e^{2x}}\,dx$ f) $\displaystyle\int_0^1 2\pi e^{2x}\sqrt{1+4e^{4x}}\,dx$

Solution: f)

6. Find the sum of $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$.

a) $\frac{1}{28}$ b) $\frac{1}{7}$ c) $\frac{1}{4}$

d) $\frac{4}{7}$ e) $\frac{7}{4}$

Solution: b)

7. What is the interval of convergence for $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2^n}$?

a) $(-\frac{1}{2}, \frac{1}{2})$ b) $[-\frac{1}{2}, \frac{1}{2})$ c) $[-\frac{1}{2}, \frac{1}{2}]$

d) $(-2, 2)$ e) $[-2, 2)$ f) $[-2, 2]$

Solution: d)

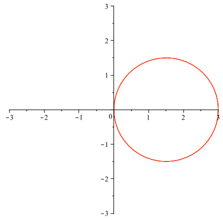
8. Find the first 4 terms of the power series for $f(x) = e^{-x^2}$ centered at 0.

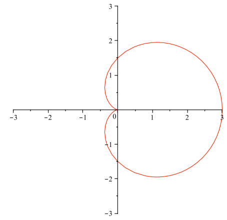
a) $1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3$ b) $1 - x + x^2 - x^3$ c) $1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6$

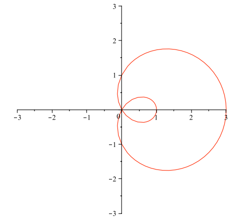
d) $1 - x^2 + x^4 - x^6$ e) $1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6$ f) $1 + x^2 + x^4 + x^6$

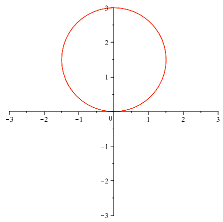
Solution: c)

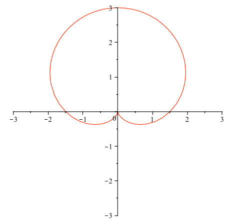
9. Which of the following is the graph of $r = 3\cos\theta$?

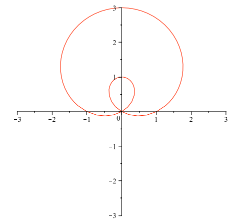
a) 

b) 

c) 

d) 

e) 

f) 

Solution: a)

Free response: Give your answer in the space provided. Answers not placed in this space will be ignored.

10. (7 points) Find the volume of the solid obtained by rotating the region bounded by $x = 1 + (y-2)^2$ and $x = 2$ about the x-axis.

Solution:

$$2 = 1 + (y-2)^2,\ (y-2)^2 = 1,\ y - 2 = \pm 1$$

so $y = 1, 3$. Shell Method:

$$\int_1^3 2\pi y(2 - (1 + (y-2)^2))\,dy = 2\pi \int_1^3 (-y^3 + 4y^2 - 3y)\,dy = 2\pi(-\frac{y^4}{4} + \frac{4}{3}y^3 - \frac{3}{2}y)|_1^3$$

$$= \frac{16\pi}{3}$$

11. (7 points) Evaluate $\int_0^{\frac{\pi}{2}} x^2 \sin x\ dx$.

Solution:

Form A Integration by parts: $u = x^2$, $du = 2x\,dx$, $dv = \sin x\,dx$, $v = -\cos x$.

$$\int_0^{\frac{\pi}{2}} x^2 \sin x\ dx = -x^2 \cos x|_0^{\pi/2} + 2\int x \cos x\,dx$$

Use integration by parts again: $u = x$, $du = dx$, $dv = \cos x\,dx$, $v = \sin x$.

$$\int_0^{\frac{\pi}{2}} x^2 \sin x\ dx = \cancel{-x^2 \cos x|_0^{\pi/2}} + 2\int x \cos x\,dx$$

$$= 2x \sin x|_0^{\pi/2} - 2\int_0^{\pi/2} \sin x\,dx = 2\frac{\pi}{2} + 2\cos(x)|_0^{\pi/2} = \pi - 2.$$

Form B

$$\frac{\pi^2}{4} - 2$$

12. (7 points) Evaluate $\int \frac{x^3}{\sqrt{x^2+1}}\,dx$.

Solution: $u = x^2 + 1$, $du = 2x\,dx$, $x^2 = u - 1$.

$$\int \frac{x^3}{\sqrt{x^2+1}}\,dx = \frac{1}{2}\int \frac{u-1}{\sqrt{u}}\,du = \frac{1}{2}\int u^{1/2} - u^{-1/2}\,du$$

$$= \frac{1}{2}(\frac{2}{3}u^{3/2} - 2u^{1/2} + C = \frac{1}{3}(1+x^2)^{3/2} - \sqrt{1+x^2} + C.$$

13. (7 points) Evaluate $\int \frac{dx}{x^3 - 2x^2 + x}$.

Solution:

$$\frac{1}{x^3 - 2x^2 + x} = \frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

$$1 = A(x-1)^2 + Bx(x-1) + Cx$$

If $x = 0$ we see $A = 1$. If $x = 1$, we see $C = 1$. If $x = -1$, then

$$1 = 1(-2)^2 + B(-1)(-2) + (-1),$$

or

$$2B = -2, \; B = -1$$

Thus,

$$\int \frac{x^3}{\sqrt{x^2+1}}\, dx = \int \frac{1}{x}\, dx - \int \frac{1}{x-1}\, dx + \int \frac{1}{(x-1)^2}\, dx$$

$$= \ln|x| - \ln|x-1| + \frac{1}{x-1} + C.$$

14. (7 points) Show whether $\int_{-1}^{2} \frac{1}{x^4}\, dx$ is convergent or divergent. If convergent, evaluate the integral.

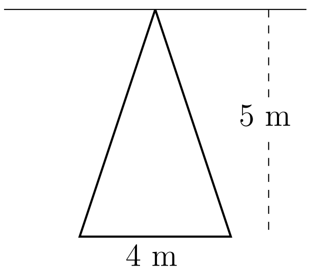
Solution:

$$\int_{-1}^{2} \frac{1}{x^4}\, dx = \int_{-1}^{0} \frac{1}{x^4}\, dx + \int_{0}^{2} \frac{1}{x^4}\, dx$$

$$\int_{0}^{2} \frac{1}{x^4}\, dx = \lim_{b \to 0^+} \int_{b}^{2} \frac{1}{x^4}\, dx = \lim_{b \to 0^+} \left(\frac{1}{24} - \frac{1}{3b^3}\right) = -\infty.$$

The integral is divergent.

15. (7 points) A vertical plate is submerged in water and has the shape shown in the figure. Find the hydrostatic force against one side of the plate. . (Use ρg to represent the weight density of water.)



Solution:

Form A If l is the distance across the plate at depth h, then similar triangles indicates that

$$\frac{l}{4} = \frac{h}{5},\ l = \frac{4}{5}h.$$

The force due to fluid pressure is

$$\int_0^5 \rho g \frac{4}{5}h \cdot h\, dh = \frac{4}{5}\rho g \int_0^5 h^2\, dh = \frac{100}{3}\rho g.$$

Form B

$$\frac{l}{5} = \frac{h}{6},\ l = \frac{5}{6}h.$$

The force due to fluid pressure is

$$\int_0^6 \rho g \frac{5}{6}h \cdot h\, dh = \frac{5}{6}\rho g \int_0^6 h^2\, dh = \frac{180}{3}\rho g.$$

16. (7 points) Determine whether the series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2 2^n}{n!}$ converges absolutely, conditionally or diverges. State which test(s) you use.

Solution: Using the ratio test, we see

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+2}\ \frac{(n+1)^2 2^{n+1}}{(n+1)!}}{(-1)^{n+1}\ \frac{n^2 2^n}{n!}}\right| = \frac{2(n+1)}{n^2} = 0$$

Since this limit is less than one, the series converges.

17. (7 points) Find the Taylor series for $f(x) = x^{-2}$ centered at $a = 1$.

Solution: $f'(x) = -2x^{-3}$, $f''(x) = -2(-3)x^{-4}$, $f'''(x) = -2(-3)(-4)x^{-5}$. $f^{(n)}(1) = (-1)^n(n+1)!$

$$f(x) = \sum_{n=0}^{\infty}(n+1)(x-1)^n$$

18. (7 points) Find the area enclosed by $r = 3 + 2\sin\theta$.

Solution:

Form A

$$\int_0^{2\pi} \frac{1}{2}(3+2\sin\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi} 9 + 12\sin\theta + 4\sin^2\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 9 + 12\sin\theta + 2 - 2\cos(2\theta)\,d\theta = \frac{1}{2}(11\theta - 12\sin\theta - \cos(2\theta))|_0^{2\pi} = 11\pi.$$

Form B

$$\int_0^{2\pi} \frac{1}{2}(4+2\sin\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi} 16 + 16\sin\theta + 4\sin^2\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 16 + 16\sin\theta + 2 - 2\cos(2\theta)\,d\theta = \frac{1}{2}(18\theta - 16\sin\theta - \cos(2\theta))|_0^{2\pi} = 18\pi.$$

END OF EXAM