Math 112 (Calculus I)
Final Exam

Multiple Choice. Fill in the answer to each problem on your computer-scored answer sheet. Make sure your name, section and instructor are on that sheet.

1. Approximate \( \int_1^5 x^4 \, dx \) using a Left Hand sum with 2 subintervals (n=2).
   (a) 82 (b) 164 (c) 81 (d) 162 (e) 624 (f) 625 (g) None of these

2. Find the area under the function \( f(x) = \sqrt[3]{x} \) from \( x = 1 \) to \( x = 8 \).
   (a) \( \frac{45}{4} \) (b) \( \frac{1}{4} \) (c) 12 (d) 15 (e) \( \frac{1}{12} \) (f) None of these

3. Given the limit statement \( \lim_{x \to 1} (2x - 3) = -1 \) pick the largest \( \delta \) that works with the definition of the limit if \( \epsilon = 0.06 \).
   (a) 0.001 (b) 0.005 (c) 0.01 (d) 0.02 (e) 0.03 (f) No such \( \delta \) exists

4. Which of the following is an inflection point of \( f(x) = \frac{x}{x^2 + 1} \)?
   (a) 1 (b) -1 (c) 2 (d) -2 (e) \( \sqrt{2} \) (f) \( -\sqrt{2} \) (g) -3 (h) \( \sqrt{3} \)

5. Given \( x \ln y - y \ln x = e^2 - 2e \), find \( \frac{dy}{dx} \) at the point \( (e^2, e) \).
   (a) 0 (b) \( e \) (c) \( e^2 \) (d) \( \frac{1-e}{e^2} \) (e) \( \frac{1-e}{e^2-2e} \) (f) \( e^2 - 2e \) (g) \( \frac{e-1}{e^2} \)

6. Which of the following are \( x \)-values for which \( f(x) = \sin(x) - x \) has a local maximum?
   (a) \(-2\pi\) (b) \(-\pi\) (c) 0 (d) \( \pi \) (e) \( 2\pi \) (f) More than one of these (g) None of these

7. Which of the following functions has a discontinuous first derivative?
   (a) \( \sinh(x) \) (b) \( x^{1/3} \) (c) \( \tan^{-1}(x) \) (d) \( \frac{x}{1+x^2} \) (e) \( \ln(x^2+1) \) (f) All of the first derivatives of these functions are continuous

8. \( \frac{d}{dx} \int_1^{2x} \sqrt{1+t^3} \, dt = \)
Solution: f)

Short Answer: Fill in the blank with the appropriate answer.

9. (11 points)

(a) Simplify \( \sin \left( \cos^{-1} \left( \frac{3}{5} \right) \right) = \frac{4}{5} \)

(b) \( \int x^2 e^{x^3} \, dx = \frac{1}{3} e^{x^3} + C \)

(c) \( \frac{d}{dx} (\ln(\sin x)) = \frac{\cos(x)}{\sin(x)} = \cot(x) \)

(d) \( \frac{d}{dx} (\sinh^2(x)) = 2 \sinh(x) \cosh(x) \) or \( \sinh(2x) \)

(e) \( \frac{d}{dx} (e^x + x^3) = e^x + 3x^2 \)

(f) If \( f'(x) = e^x + \sin x + x^2 \), then \( f(x) = e^x - \cos(x) + \frac{x^3}{3} + C \)

(g) \( \lim_{x \to \infty} \tan^{-1}(x) = \frac{\pi}{2} \)

(h) \( \lim_{x \to 2} \frac{x + 2}{x^2 - 4} = \) Does not exist

(i) \( \lim_{x \to 0^+} \sin x - \ln x = \infty \)

(j) \( \int_1^2 4 + 5x \, dx = \frac{23}{2} \)

(k) \( \frac{d}{dx} (2^x) = \ln(2)2^x \)
10. (8 points)

(a) If \( f(x) = x^{-1} \), use the definition of a derivative to set up a limit to find \( f'(x) \).

Solution:

\[
f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}
\]

(b) Find \( f'(x) \) by evaluating the limit. (No points will be awarded if differentiation rules are used.)

Solution:

\[
f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{-h}{h(x+h)x}
\]

\[
= \lim_{h \to 0} \frac{-1}{(x+h)x} = -\frac{1}{x^2}.
\]

11. (6 points) Find the dimension of the largest rectangle that can be inscribed between the curve \( y = 4 - x^2 \) and the \( x \)-axis.

Solution:

Form A:

The rectangle will have corners at \((-x, 0), (x, 0), (x, 4-x^2), \) and \((-x, 4-x^2)\). Thus, the width of the rectangle is \(2x\) and the height is \(4-x^2\). The area therefore is \(A(x) = 2x(4-x^2) = 8x - 2x^3\).

Notice that

\[A'(x) = 8 - 6x^2.\]

If we set the derivative to 0, we have \(x^2 = \frac{4}{3}\), or \(x = \frac{2}{\sqrt{3}}\). Thus, the dimensions are \((\frac{4}{\sqrt{3}}, \frac{8}{3})\).

Form B:

Here the area is \(A(x) = 2x(9-x^2) = 18x - 2x^3\). \(A'(x) = 18 - 6x^2 = 0\) gives \(x^2 = 3\), so \(x = \sqrt{3}\). Thus, the dimensions are \((\sqrt{3}, 6)\).

12. (4 points) \( \lim_{x \to 0} \ln(x) \sin(x) \)

Solution:

Form A:

\[\lim_{x \to 0} \ln(x) \sin(x) = \lim_{x \to 0} \frac{\ln x}{\csc x}.
\]

Use L'Hopital’s rule:

\[\lim_{x \to 0} \frac{\ln x}{\csc x} = \lim_{x \to 0} \frac{\frac{1}{x}}{-\csc x(\cot x)} = \lim_{x \to 0} \frac{\sin x \tan x}{x}
\]

\[= \lim_{x \to 0} \cos x \tan x + \sin x \sec^2 x = 0.
\]

Form B:

\[\lim_{x \to 0} \ln(x)(1 - \cos(x)) = \lim_{x \to 0} \frac{1 - \cos(x)}{\ln x}
\]

\[= \lim_{x \to 0} \frac{\sin(x)}{\frac{1}{x \ln x}} = \frac{\cos(x)}{\ln^2 x + 2 \ln x} = 0.
\]
13. (3 points) \( \frac{d}{dx} \left( \ln \left( xe^x - \frac{\sin x}{x} \right) \right) \)

Solution:
\[
\frac{1}{(xe^x - \frac{\sin x}{x})} \left( e^x + xe^x - \frac{x \cos x - \sin x}{x^2} \right)
\]

14. (4 points) \( \int \frac{x}{x^2 + 4} \, dx \)

Solution:
Form A:
Let \( u = x^2 + 4 \). Then \( du = 2x \, dx \) and
\[
\int \frac{x}{x^2 + 4} \, dx = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C
\]
\[
= \frac{1}{2} \ln (x^2 + 4) + C.
\]

Form B:
Let \( u = x^3 + 9 \). Then \( du = 3x^2 \, dx \) and
\[
\int \frac{x^2}{x^3 + 9} \, dx = \frac{1}{3} \int \frac{du}{u} = \frac{1}{3} \ln |u| + C
\]
\[
= \frac{1}{3} \ln |x^3 + 9| + C.
\]
15. (10 points) Give the following information about the function \( f(x) = x^4 - 4x^3 \): (If no information is available in a particular category, leave it blank or cross it out. Putting information in where none exists will be treated as an incorrect answer).

Form A:
All \( x \)-intercepts = \((0, 0)\) \((4, 0)\)
y-intercept =\((0, 0)\)
Intervals for which \( f(x) \) is increasing: \((3, \infty)\)
Intervals for which \( f(x) \) is decreasing: \((-\infty, 3)\)
Coordinates of all inflection points: \((0, 0), (2, -16)\)
Intervals for which \( f(x) \) is concave up: \((-\infty, 0), (2, \infty)\)
Intervals for which \( f(x) \) is concave down: \((0, 2)\)
Coordinates of any local maximums: None
Coordinates of any local minimums: \((3, -27)\)

Form B:
All \( x \)-intercepts = \((0, 0)\) \((5, 0)\)
y-intercept =\((0, 0)\)
Intervals for which \( f(x) \) is increasing: \((-\infty, 0), (4, \infty)\)
Intervals for which \( f(x) \) is decreasing: \((0, 4)\)
Coordinates of all inflection points: \((3, -162)\)
Intervals for which \( f(x) \) is concave up: \((3, \infty)\)
Intervals for which \( f(x) \) is concave down: \((-\infty, 0), (0, 3)\)
Coordinates of any local maximums: None
Coordinates of any local minimums: \((4, -256)\)

16. (6 points) A certain element has a half life of 20 years. How many years will it take until only 10% of the element remains? (Note: \( \ln \left( \frac{1}{2} \right) \approx -0.693 \) and \( \ln \left( \frac{1}{10} \right) \approx -2.3026 \). You can either leave your answer in terms of logs or give a numerical answer using these approximations.)

Solution: Since \( \frac{1}{2} = e^{20r} \), it is easy to see that \( r = \frac{\ln(1/2)}{20} = -\frac{\ln 2}{20} \). For 10 percent to remain, we need
\[
.1 = e^{-\frac{\ln 2}{20}t},
\]
or
\[
\ln(.1) = -\frac{\ln 2}{20}t.
\]
Hence,
\[
t = -\frac{\ln 2}{20 \ln(.1)} = \frac{\ln 2}{20 \ln(10)}.
\]
17. (6 points) The equation of the tangent line to the curve \( y = \frac{1}{x^2} \) at \( (2, \frac{1}{4}) \).

**Solution**:  
Form A:  
Note that \( y' = \frac{-2}{x^3} \). So, \( y'(2) = \frac{-1}{4} \). Thus,  
\[
y - \frac{1}{4} = \frac{1}{4}(x - 2),
\]
and  
\[
y = \frac{1}{4}x + \frac{3}{4}.
\]
Form B:  
y' = \frac{-3}{x^4}. y'(\frac{1}{2}) = -48.  
\[
y - 8 = -48(x - \frac{1}{2})
\]
or  
\[
y = -48x + 32.
\]

18. (6 points) Use linear approximation to estimate \( \sqrt{63} \):

**Solution**: Let \( f(x) = \sqrt{x} \). Then \( f'(x) = \frac{1}{2\sqrt{x}} \).

\[
\sqrt{63} = \sqrt{64} + \frac{1}{2\sqrt{64}}(-1)
\]
\[
8 - \frac{1}{16} = \frac{127}{16}.
\]

19. (6 points) A pump is blowing up a spherical balloon with a pump rate of 10cm\(^3\)/sec. How fast is the diameter of the balloon growing when the balloon has a 5cm radius? (Volume of a sphere is given by \( \frac{4}{3}\pi r^3 \).)

**Solution**:  
Form A:  
Differentiate \( V = \frac{4}{3}\pi r^3 \) to get  
\[
\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.
\]
Thus,  
\[
10 = 4\pi \cdot 25 \frac{dr}{dt},
\]
and  
\[
\frac{dr}{dt} = \frac{1}{10\pi}.
\]
Form B:  
10 = 4\pi \cdot 9 \frac{dr}{dt}  
\[
\frac{dr}{dt} = \frac{5}{18\pi}.
\]
20. (6 points) A particle is moving with the given data. Find the position function of the particle.

\[ a(t) = \sin(t) + 3 \cos(t), \quad s(0) = 0, \quad v(0) = 2. \]

**Solution:**

\[ v(t) = -\cos(t) + 3 \sin(t) + C \]

\[ v(0) = -1 + C = 2 \]

so \( C = 3 \). Thus,

\[ v(t) = -\cos(t) + 3 \sin(t) + 3. \]

\[ s(t) = -\sin(t) - 3 \cos(t) + 3t + D \]

\[ s(0) = -3 + D = 0, \quad D = 3 \]

\[ s(t) = -\sin(t) - 3 \cos(t) + 3t + 3. \]