PH. D. QUALIFYING EXAM SPRING 2009 - ALGEBRA

Answer all of the questions.

1. Let $G$ be a group of odd order and let $x \in G$. Show that if $x$ is conjugate to $x^{-1}$, then $x$ is the identity element.

2. Let $G$ be a finite group and let $p$ be a prime. Write $|G| = p^m a$, where $\gcd(p, a) = 1$. Prove that $G$ has a subgroup of order $p^m$. Such a subgroup is called a Sylow $p$ subgroup of $G$.

3. Let $G$ be a finite abelian group. Prove the equivalence of:
   (1) $G$ is cyclic;
   (2) Every subgroup of $G$ is cyclic.

4. Determine (up to isomorphism) the possible group structures on a set with 6 elements.

5. Let $A$ be a commutative ring with identity $1 \neq 0$. Determine (up to isomorphism) all the possibilities for $A$ if $|A| = 4$.

6. Let $T : V \to W$ be a linear transformation of vector spaces of dimension $n, m$ (possibly infinite). Show that
   \[ \dim \ker T + \dim \text{Im}(T) = n. \]
   Here $\text{Im}(T)$ is the image of $T$ and $\ker T$ is the kernel of $T$.

7. What is the Galois group of the polynomial $x^4 + 25 \in \mathbb{Q}[x]$ (it will suffice to list generators for the Galois group and then determine its isomorphism class).

8. Let $R$ be a ring with identity and let $e \in R$ be an idempotent, so that $e^2 = e$. Assume that $e$ is central, so that $er = re$ for all $r \in R$. Show that $eR$ and $(1-e)R$ are 2-sided ideals of $R$ and that $R \cong eR \times (1-e)R$.

9. Let $R$ be a commutative ring with identity $1 \neq 0$.
   a) Show that any proper ideal of $R$ is contained in a maximal ideal.
   b) Show that $R$ has a quotient which is a field.

10. State and prove Eisenstein’s criterion for irreducibility of polynomials over $\mathbb{Z}$.