Ph.D. QUALIFIER EXAMINATION: ANALYSIS
Fall 2009

Instructions: Answer exactly 6 of the 10 questions given. If you answer more than 6 questions, your grade will be determined by the first 6 questions that you answered.

Some Notation.
1. $\mathbb{R}^k$ – Euclidean $k$-dimensional space
2. $\mathbb{C}$ – the complex numbers
3. $(X, \mathcal{M}, \mu)$ – a measure space where $X$ is a set, $\mathcal{M}$ is a $\sigma$-algebra of subsets of $X$, and $\mu$ is a measure on $\mathcal{M}$
4. a.e.$[\mu]$ – almost every with respect to the measure $\mu$
5. $m$ – Lebesgue measure on $\mathbb{R}^k$
6. $\|f\|_p = \left( \int_X |f|^p \, d\mu \right)^{1/p}$ – the $L^p$-norm of a $\mu$-measurable function $f : X \to \mathbb{C}$
7. $\|f\|_\infty$ – the essential supremum of $f$
8. $p, q$ – conjugate exponents where $\frac{1}{p} + \frac{1}{q} = 1$
9. $L^p(\mu)$ – the space of $\mu$-measurable functions $f : X \to \mathbb{C}$ with $\|f\|_p < \infty$
10. $L^p(\mathbb{R}^k)$ – the space of Lebesgue measurable functions $f : \mathbb{R}^k \to \mathbb{C}$ with $\|f\|_p < \infty$
11. $\|\Gamma\| = \sup\{\|\Gamma x\| : x \in X, \|x\| \leq 1\}$ – operator norm of a linear transformation $\Gamma : X \to Y$ where $X$ and $Y$ are normed linear spaces
12. $|\lambda|$ – the total variation of a measure $\lambda$.
13. $\lambda \ll \mu$ – the measure $\lambda$ is absolutely continuous with respect to the measure $\mu$
14. $\lambda \perp \mu$ – the measures $\lambda$ and $\mu$ are mutually singular
15. $\frac{d\lambda}{d\mu}$ – the Radon-Nikodym derivative of $\lambda$ with respect to $\mu$ where $\lambda \ll \mu$
16. Lip $\alpha$ – the space of complex functions $f$ on $[a, b]$ for which $\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty$; here $0 < \alpha \leq 1$
17. $f * g$ – the convolution of $f$ and $g$: $(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - y)g(y) \, dy$
18. $C_c(\mathbb{R}^k)$ – the continuous complex functions on $\mathbb{R}^k$ whose support is compact.
19. $\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ixt} \, dx$ – the Fourier transform
Questions

1. State and prove Fatou’s Lemma. [You may assume the Monotone Convergence Theorem in your proof.]

2. Let $A$ be a compact subset of $\mathbb{R}^k$. Prove that if $f$ is a bounded complex Lebesgue measurable function on $\mathbb{R}^k$ whose support is a subset of $A$, then for each $\epsilon > 0$ there exists $g \in C_c(\mathbb{R}^k)$ such that $m(\{x \in \mathbb{R}^k : f(x) \neq g(x)\}) < \epsilon$.

3. Let $S$ be the class of all complex, measurable, simple functions on $X$ such that $\mu(\{x : s(x) \neq 0\}) < \infty$. If $1 \leq p < \infty$, prove that $S$ is dense in $L^p(\mu)$.

4. Find a nonempty closed subset in $L^2(T)$ that contains no element of smallest norm. [Recall that $L^2(T)$ is the Hilbert space of all complex, Lebesgue measurable $2\pi$-periodic functions on $\mathbb{R}$.]

5. State and prove the Banach-Steinhaus Theorem. [You may assume Baire’s Theorem in your proof.

6. Let $L^\infty = L^\infty(m)$ where $m$ is Lebesgue measure on $I = [0, 1]$. Show that there is a bounded linear functional $\Lambda \neq 0$ on $L^\infty$ that is 0 on $C(I)$, the space of continuous functions on $I$. Show further that there is no $g \in L^1(m)$ such that $\Lambda(f) = \int_I fg \, dm$ for all $f \in L^\infty$. [You may assume the Hahn-Banach Theorem in your proof.] 

7. For a complex measure $\mu$, prove that

$$|\mu|(E) = \sup \left\{ \left| \int_E f \, d\mu \right| : f \text{ is measurable and } |f| \leq 1 \right\}$$

for every $E \in \mathcal{M}$.

8. If $f \in \text{Lip } 1$ on $[a, b]$, prove that $f$ is absolutely continuous and that $f' \in L^\infty$.

9. For $n = 1, 2, 3, \ldots$, let $g_n$ be the characteristic function of $[-n, n]$. Compute $g_n \ast g_1$ explicitly. Prove there is a function $f_n \in L^1$ such that

$$g_n \ast g_1 = \hat{f}_n.$$

Show that $\|f_n\|_1 \to \infty$ as $n \to \infty$.

10. State and prove the Residue Theorem. [You may assume Cauchy’s Theorem in your proof.]