Weyl’s theorem in the measure theory of numbers

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Dedicated to the memory of Walter Philipp

Abstract Let $S = \{a_1, a_2, \ldots \}$ be a real sequence,

$$a_{k+1} - a_k \geq \sigma > 0 \quad (k = 1, 2, \ldots).$$

Weyl proved in 1916 that the sequence

$$Sx : a_1x, a_2x, a_3x, \ldots$$

is uniformly distributed (mod 1) for almost all $x \in \mathbb{R}$. Interpreted in the obvious way, this remains valid if $x$ varies over $\mathbb{R}^d$. The following results are proved about the null set

$$E^{(d)}(S) = \{x \in \mathbb{R}^d : Sx \text{ is not uniformly distributed (mod 1)}\}.$$

(i) If $a_k = O(k^p)$, then $E^{(d)}(S)$ has dimension $\leq d - 1/p$. Given $p$, this bound is attained for suitable $S$.

(ii) The intersection of $E^{(d)}(S)$ with a curve $C$ satisfying natural conditions is a null subset of $C$, and has dimension $\leq 1 - 1/pd$ if $a_k = O(k^p)$.

(iii) Let $d = 1$. Suppose $a_k \in \mathbb{N}$ $(k \geq 1)$, $a_k \leq Ck$ for infinitely many $k$. The subset $B_{\delta}(S)$ of $E^{(d)}(S) \cap [0,1)$ consisting of $x$ for which $Sx$ has bias $b(x) \geq \delta > 0$ (defined below) is finite. A cardinality bound is given, and is strengthened for the set

$$H_I(S) = \{x : Sx \text{ omits the interval } I \text{ (mod 1)}\}.$$
Now
\[ \sum_{|s| \leq CLN} |d_s|^2 \leq \frac{9}{4} \beta^2 \sum_{|s| \leq CLN} f_s^2. \]
This last sum is simply \(2M\), where \(M\) is the number of solutions to
\[ \lambda_k = ma, \quad 1 \leq l, m \leq L, \ 1 \leq k, r \leq N. \]
A trivial bound for \(M\) is \(NL^2\). We can obtain a different bound by noting that for fixed \(l, m\) we must have \(a_k \equiv 0 \pmod{m/(m,l)}\) and there are \(\leq CN(m,l)/m\) solutions to this. This yields
\[ M \leq CN \sum_{1 \leq l, m \leq L} \frac{(m,l)}{m}, \]
\[ \leq CN \sum_{d \leq L} d \left( \frac{L}{d} \right) \sum_{m \leq L/d} \frac{1}{md}, \]
\[ \leq CNL \sum_{d \leq L} \frac{1}{d} \sum_{m \leq L/d} \frac{1}{m} \]
\[ \leq CNL \left( \log(eL) \right)^2, \]
and
\[ \sum_{|s| \leq CLN} |d_s|^2 \leq \frac{9}{2} \beta^2 \min \left( NL^2, CNL \left( \log(eL) \right)^2 \right). \]
As for the other factor on the left-hand side of (4.10), we have
\[ \sum_{|s| \leq CLN} \left| \sum_{i=1}^u e(sx_i) \right|^2 \leq (2 + \varepsilon) CLNu \]
for sufficiently large \(N\), by Lemma 7. We conclude that
\[ 18(2 + \varepsilon) Cu(LN)^2 \left( L, C \left( \log(eL) \right)^2 \right) \geq (Nu)^2. \]
Since \(\varepsilon\) is arbitrary, this gives the stated result. \(\square\)

References

