Instructions: Give solutions to exactly 6 of the following 8 problems. If you give more than 6 solutions, your grade will be determined by the first six that appear.

(1) Answer the following.
   (a) Define a hyperbolic invariant set.
   (b) Define an $\epsilon$-chain.
   (c) State the Shadowing Lemma.
   (d) Assume that $\mathcal{R}(f)$ (the chain recurrent set) is hyperbolic. Prove that the periodic points are dense in $\mathcal{R}(f)$.

(2) Define rotation number for a circle homeomorphism and show that the rotation number for an orientation preserving circle homeomorphism is rational if and only if there exists a periodic point.

(3) Let $f$ be a diffeomorphism and $p$ a periodic point for $f$. Let $H_p$ be the equivalence class of periodic points heteroclinically related to $p$, i.e. $q \sim p$ if $W^s(p) \cap W^u(q) \neq \emptyset$ and $W^s(q) \cap W^u(p) \neq \emptyset$. Let $\Lambda_p$ be the closure of $H_p$. Prove that $\Lambda_p$ is transitive.

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 5x(1-x)$. Describe, up to topological conjugacy, the dynamics of the set of points $x$ such that $f^n(x) \in [0,1]$ for all $n \geq 0$. Prove your answer.

(5) Let $x$ be a recurrent point of a $C^1$ 2-d system: $x' = f(x)$. Prove that either $x$ is an equilibrium point or $x$ lies on a closed orbit.

(6) Consider a differential equation $x' = Ax + f(x)$, where $x \in \mathbb{R}^n$, $A$ is a $n \times n$ matrix with a positive eigenvalue, $f$ is a $C^1$ function from $\mathbb{R}^n$ to $\mathbb{R}^n$ with $f(0) = 0$ and $f'(0) = 0$. Prove that $x = 0$ is unstable.

(7) Let $f$ be a $C^1$ vector field on a neighborhood of the annulus $A = \{x \in \mathbb{R} \mid 1 \leq |x| \leq 2\}$. Suppose that $f$ has no zeros and $f$ is transverse to the boundary, pointing inward. Prove there is a closed orbit for $x' = f(x)$.

(8) Consider $x' = f(x)$ and its perturbed equation $x' = f(x) + \epsilon h(t,x)$ where $x \in \mathbb{R}^n$, $f$ and $h$ are $C^2$ functions, and $h(t,x)$ is $T$-periodic in $t$, $0 < \epsilon$ is a parameter. Prove that if $x' = f(x)$ has a hyperbolic equilibrium point $p^*$, then the perturbed equation $x' = f(x) + \epsilon h(t,x)$ has a unique periodic solution $p(t,\epsilon)$ such that $p(t,\epsilon) - p^* = O(\epsilon)$. 

1