Algebraic topology qualifying exam February 2000

Answer eight questions, four from part I and four from part II. Give as much detail in your answers as you can.

Part I

1. a) Let $G$ and $H$ be functors from a category $C$ to a category $D$. Define a natural transformation from $G$ to $H$. b) For an admissible pair of topological spaces $(X,A)$ define functors $G$ and $H$ by $G(X,A) = H_p(X,A)$, $H(X,A) = H_{p-1}(X,A)$. Show that the map $\partial^*$ is a natural transformation of $G$ to $H$. Define and give an example (with proof) of a contravariant functor.

2. State and prove the Kunneth theorem for topological spaces.

3. a) Let $F$ be the closed orientable surface of genus 2. Find a presentation for the fundamental group of $F$. b) Prove that $\pi_{n}(X)$ is abelian for $n > 1$.

4. State and prove Poincaré duality for orientable triangulated compact homology $n$-manifolds.

5. a) Give the definition of cap products ($\cap$). b) Show that, for $K$ a simplicial complex and coefficients $R$, it gives a homomorphism $H_p(K,R) \otimes H_{p+q}(K,R) \to H_q(K,R)$.

Part II

6. The cyclic group $G = \mathbb{Z}/p\mathbb{Z}$ ($p$ a prime) acts on the unit sphere $S^{2n-1}$ as follows: let $z = \exp(2\pi i/p)$ be a $p$th root of unity and think of $G$ as the subgroup of the complex numbers generated by $z$ under multiplication. Then the action is: $z(x_1, \ldots, x_{2n}) = (zx_1, \ldots, zx_{2n})$. Let $L^n = S^{2n-1}/(Z/p\mathbb{Z})$. Why is $L^n$ a manifold? Find $H_*(L^n, \mathbb{Z})$ (where $Z = \text{integer ring}$).

7. a) Find $H_*(S^n \times S^n)$. b) Find $H_*(S^n \vee S^n \vee S^{n+m})$. What do you notice?

8. Let $M$ be a connected compact $n$-manifold with boundary $B$ where $n > 1$. a) show that $B$ is not a retract of $M$. b) Prove that if $M$ is contractible, then $B$ has the homology of a sphere.

9. Let $T = S^1 \times S^1 \times S^3$. Find the cohomology groups of $T$ and its ring structure with coefficients the integers.

10. a) What are $(\mathbb{Z}/m\mathbb{Z})^\ast \ast \mathbb{Z}$ and $\text{Ext}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})$. Here * denotes the torsion product and you must give a proof in each case. $Z$ denotes the integers. b) Find $H_*(S^1 \times RP^2, \mathbb{Z}/m\mathbb{Z})$. 