**Ph.D. QUALIFIER EXAMINATION: ANALYSIS**

Fall 2004

**Instructions:** Answer *exactly* 6 of the 10 questions given. If you do more than 6 questions, your grade will be determined by the first 6 questions that you answered.

**Some Notation.**

1. $\mathbb{R}^k$ – Euclidean $k$-dimensional space
2. $\mathbb{C}$ – the complex numbers
3. $(X, \mathcal{M}, \mu)$ – a measure space where $X$ is a set, $\mathcal{M}$ is a $\sigma$-algebra of subsets of $X$, and $\mu$ is a measure on $\mathcal{M}$
4. a.e.$[\mu]$ – almost every with respect to the measure $\mu$
5. $m$ – Lebesgue measure on $\mathbb{R}^k$
6. $\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{1/p}$ – the $L^p$-norm of a $\mu$-measurable function $f : X \to \mathbb{C}$
7. $\|f\|_\infty$ – the essential supremum of $f$
8. $p, q$ – conjugate exponents where $\frac{1}{p} + \frac{1}{q} = 1$
9. $L^p(\mu)$ – the space of $\mu$-measurable functions $f : X \to \mathbb{C}$ with $\|f\|_p < \infty$
10. $L^p(\mathbb{R}^k)$ – the space of Lebesgue measurable functions $f : \mathbb{R}^k \to \mathbb{C}$ with $\|f\|_p < \infty$
11. $|\lambda|$ – the total variation of a measure $\lambda$.
12. $\lambda \ll \mu$ – the measure $\lambda$ is absolutely continuous with respect to the measure $\mu$
13. $\lambda \perp \mu$ – the measures $\lambda$ and $\mu$ are mutually singular
14. $\frac{d\lambda}{d\mu}$ – the Radon-Nikodym derivative of $\lambda$ with respect to $\mu$ where $\lambda \ll \mu$
15. Lip $\alpha$ – the space of complex functions $f$ on $[a, b]$ for which $\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty$; here $0 < \alpha \leq 1$
16. $f * g$ – the convolution of $f$ and $g$: $(f * g)(x) = \int_{-\infty}^{\infty} f(x - y) g(y) \, dy$
17. $C_0(\mathbb{R})$ – the continuous complex functions on $\mathbb{R}$ which vanish at infinity
18. $\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-i xt} \, dm(x)$ – the Fourier transform
1. State and prove Lebesgue’s Dominated Convergence Theorem. [You may assume Fatou’s Lemma in your proof.]

2. Construct a sequence of continuous functions \( f_n \) on \([0, 1]\) such that \( 0 \leq f_n \leq 1 \) and
\[
\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = 0,
\]
but the sequence \( \{f_n(x)\} \) does not converge for any \( x \in [0, 1] \).

3. Suppose \( 1 \leq p < q < r \leq \infty \). Prove that if \( f \in L^p(\mu) \cap L^r(\mu) \), then \( f \in L^q(\mu) \).

4. Suppose that \( X \) and \( Y \) are Banach spaces. Suppose that \( \Lambda : X \to Y \) is a linear mapping with the property that for every sequence \( \{x_n\} \) in \( X \) such that \( x = \lim x_n \) and \( y = \lim \Lambda x_n \) exist, it follows that \( y = \Lambda x \). Prove that \( \Lambda \) is continuous. [You may assume that a continuous, one-to-one linear mapping from one Banach space onto another Banach space has an inverse that is a continuous linear mapping.]

5. Let \( \{f_n\} \) be a sequence of continuous complex functions on a nonempty complete metric space \( X \) such that \( f(x) = \lim f_n(x) \) exists for every \( x \in X \) (i.e. \( f_n \to f \) pointwise). Prove for every \( \epsilon > 0 \) there is a nonempty open set \( V \) and a positive integer \( N \) such that \( |f(x) - f_n(x)| \leq \epsilon \) whenever \( x \in V \) and \( n \geq N \).

6. Suppose that \( \mu \) and \( \lambda \) are measures on a \( \sigma \)-algebra \( \mathcal{M} \) with \( \mu \) positive and \( \lambda \) complex. Prove that \( \lambda \ll \mu \) if and only if for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( |\lambda(E)| < \epsilon \) for all \( E \in \mathcal{M} \) with \( \mu(E) < \delta \).

7. Let \( \mu \) be a complex Borel measure on \( \mathbb{R}^k \). Define the symmetric derivative, \( D\mu \), of \( \mu \) with respect to \( m \). Define a Lebesgue point of an \( L^1(\mathbb{R}^k) \) function. Prove that if \( \mu \ll m \) and \( f \) is the Radon-Nikodym derivative of \( \mu \) with respect to \( m \), then
\[
D\mu = f \text{ a.e.} \, [m], \quad \text{and} \quad \mu(E) = \int_E (D\mu) \, dm.
\]
[You may assume that almost every \( x \in \mathbb{R}^k \) is a Lebesgue point of an \( L^1(\mathbb{R}^k) \) function.]

8. Suppose \( p \) and \( q \) are conjugate exponents with \( 1 < p < \infty \), and set \( \alpha = 1/q \). Prove that if \( f \) is absolutely continuous on \([a, b]\) and \( f' \in L^p \), then \( f \in \text{Lip} \, \alpha \).

9. Prove that if \( f, g \in L^1(\mathbb{R}) \), then \( f \ast g \) is \( L^1(\mathbb{R}) \) with \( \|f \ast g\|_1 \leq \|f\|_1 \|g\|_1 \).

10. Prove that if \( f \in L^1(\mathbb{R}) \), then \( \hat{f} \in C_0(\mathbb{R}) \) and \( \|\hat{f}\|_\infty \leq \|f\|_1 \). [You may assume for each \( x \in \mathbb{R} \) that the map \( y \to f(x - y) \) from \( \mathbb{R} \) to \( L^1(\mathbb{R}) \) is uniformly continuous.]