Ph.D. QUALIFIER EXAMINATION: ANALYSIS
Winter 2005

Instructions: Answer exactly 6 of the 10 questions given. If you do more than 6 questions, your grade will be determined by the first 6 questions that you answered.

Some Notation.

1. $\mathbb{R}^k$ – Euclidean $k$-dimensional space
2. $\mathbb{C}$ – the complex numbers
3. $(X, \mathcal{M}, \mu)$ – a measure space where $X$ is a set, $\mathcal{M}$ is a $\sigma$-algebra of subsets of $X$, and $\mu$ is a measure on $\mathcal{M}$
4. a.e.$[\mu]$ – almost every with respect to the measure $\mu$
5. $m$ – Lebesgue measure on $\mathbb{R}^k$
6. $\|f\|_p = \left( \int_X |f|^p \, d\mu \right)^{1/p}$ – the $L^p$-norm of a $\mu$-measurable function $f : X \to \mathbb{C}$
7. $\|f\|_\infty$ – the essential supremum of $f$
8. $L^p(\mu)$ – the space of $\mu$-measurable functions $f : X \to \mathbb{C}$ with $\|f\|_p < \infty$
9. $L^p(\mathbb{R}^k)$ – the space of Lebesgue measurable functions $f : \mathbb{R}^k \to \mathbb{C}$ with $\|f\|_p < \infty$
10. $|\lambda|$ – the total variation of a measure $\lambda$.
11. $\lambda \ll \mu$ – the measure $\lambda$ is absolutely continuous with respect to the measure $\mu$
12. $\lambda \perp \mu$ – the measures $\lambda$ and $\mu$ are mutually singular
13. $\frac{d\lambda}{d\mu}$ – the Radon-Nikodym derivative of $\lambda$ with respect to $\mu$ where $\lambda \ll \mu$
14. $\text{Lip } \alpha$ – the space of complex functions $f$ on $[a, b]$ for which $\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty$; here $0 < \alpha \leq 1$
15. $f * g$ – the convolution of $f$ and $g$: $(f * g)(x) = \int_{-\infty}^{\infty} f(x - y)g(y) \, dy$
16. $\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-ixt} \, dm(x)$ – the Fourier transform
1. Suppose that \( f \in L^1(\mu) \). Prove that for each \( \epsilon > 0 \) there is \( \delta > 0 \) such that
\[
\int_E |f| \, d\mu < \epsilon
\]
whenever \( \mu(E) < \delta \).

2. Let \( X = [0,1] \). Prove that if \( f \) is a complex Lebesgue measurable on \( X \) with \( |f| \leq 1 \), then there exists a sequence \( \{g_n\} \) such that \( g_n \in C([0,1]) \), \( |g_n| \leq 1 \), and
\[
f(x) = \lim_{n \to \infty} g_n(x) \text{ a.e.}[m].
\]
[You may assume Lusin’s Theorem in your proof.]

3. If \( \mu(X) < \infty \) and \( 0 < p < q \leq \infty \), prove that \( L^p(\mu) \supset L^q(\mu) \).

4. Let \( H \) be a Hilbert space with inner product \((\cdot, \cdot)\). Prove that if \( L \) is a continuous linear functional on \( H \), then there exists a unique \( y \in H \) such that
\[
Lx = (x, y) \text{ for all } x \in H.
\]

5. Let \( \{f_n\} \) be a sequence of continuous complex functions on a nonempty complete metric space \( X \) such that \( f(x) = \lim_{n \to \infty} f_n(x) \) exists for every \( x \in X \) (i.e. \( f_n \to f \) pointwise). Prove that there is a nonempty open set \( V \) and a number \( M < \infty \) such that
\[
|f_n(x)| < M \text{ for all } x \in V \text{ and for all } n = 1, 2, 3, \ldots.
\]

6. Suppose that \( \mu, \lambda, \lambda_1, \text{ and } \lambda_2 \) are measures on a \( \sigma \)-algebra \( M \) with \( \mu \) positive. Prove the following:
   (a) If \( \lambda_1 \perp \mu \) and \( \lambda_2 \perp \mu \), then \( \lambda_1 + \lambda_2 \perp \mu \);
   (b) If \( \lambda \ll \mu \), then \( |\lambda| \ll \mu \).

7. Let \( \mu \) be a complex Borel measure on \( \mathbb{R}^k \). Define the symmetric derivative, \( D\mu \), of \( \mu \) with respect to \( m \). Define a Lebesgue point of an \( L^1(\mathbb{R}^k) \) function. Prove that if \( \mu \ll m \) and \( f \) is the Radon-Nikodym derivative of \( \mu \) with respect to \( m \), then
\[
D\mu = f \text{ a.e.}[m], \quad \text{and} \quad \mu(E) = \int_E (D\mu) \, dm.
\]
[You may assume that almost every \( x \in \mathbb{R}^k \) is a Lebesgue point of an \( L^1(\mathbb{R}^k) \) function.]

8. If \( f \in \text{Lip} 1 \) on \( [a, b] \), prove that \( f \) is absolutely continuous and that \( f' \in L^\infty \).

9. Suppose that \( (X, \mathcal{G}, \mu) \) and \( (Y, \mathcal{H}, \lambda) \) are \( \sigma \)-finite measure spaces, and suppose that \( \psi \) is a measure on \( \mathcal{G} \times \mathcal{H} \) such that
\[
\psi(A \times B) = \mu(A)\lambda(B)
\]
for all \( A \in \mathcal{G} \) and all \( B \in \mathcal{H} \). Prove that \( \psi(E) = (\mu \times \lambda)(E) \) for all \( E \in \mathcal{G} \times \mathcal{H} \). [You may assume that \( \mathcal{G} \times \mathcal{H} \) is the smallest monotone class which contains all elementary sets.]

10. Prove that if \( f \) and \( g \) are \( L^1(\mathbb{R}) \) and \( h = f * g \), then \( \hat{h}(t) = \hat{f}(t)\hat{g}(t) \). [You may assume that \( h \) is \( L^1(\mathbb{R}) \).]