Phd Exam Winter 2006

Work at 7 problems from the real analysis section and 3 from the complex analysis section.

Real Analysis

1. Let \( \{ \phi_k \}_{k=1}^{\infty} \) be a sequence of continuous functions such that \( \int \phi_k(x) \, dm_n = 1 \), \( \phi_k(x) \geq 0 \) for all \( x \), and \( \phi_k \) equals zero for all \( |x| > 1/k \). Now suppose \( f \in L^p(\mathbb{R}^n) \). Show \( \lim_{k \to \infty} ||f * \phi_k - f||_{L^p(\mathbb{R}^n)} = 0 \).

2. Let \((\Omega, \mathcal{F}, \mu)\) be a measure space and suppose \( \{ f_n \} \) is a sequence of measurable functions mapping \( \Omega \) to \( \mathbb{R} \). Show the set on which this sequence converges must be measurable.

3. Suppose \( \mathcal{F} \) is a uniformly integrable set of functions defined on a measure space, \((\Omega, \mathcal{F}, \mu)\). Let \( |\mathcal{F}| = \{ |f| : f \in \mathcal{F} \} \). Show that \( |\mathcal{F}| \) is also uniformly integrable.

4. Does there exist a strictly increasing function defined on \([0, 1]\) which has a derivative a.e. and this derivative equals zero on a set of positive measure? If there is, you need to describe one. If there isn’t you need to explain why there isn’t one.

5. The Maximal function, \( Mf \) for \( f \in L^1_{loc}(\mathbb{R}^n) \) was defined as

\[
Mf(x) \equiv \sup_{r>0} \frac{1}{m_n(\mathcal{B}(x,r))} \int_{\mathcal{B}(x,r)} |f| \, dm_n
\]

Show that \( Mf \) is Borel measurable. **Hint:** You might first show

\[
x \rightarrow f_r(x) = \frac{1}{m_n(\mathcal{B}(x,r))} \int_{\mathcal{B}(x,r)} |f| \, dm_n
\]

is continuous.

6. If \( f : \mathbb{R}^n \rightarrow [0, \infty] \) is Lebesgue measurable, show there exists \( g : \mathbb{R}^n \rightarrow [0, \infty] \) such that \( g = f \) a.e. and \( g \) is Borel measurable.

7. It can be proved that for \( f \) an increasing continuous function defined on an interval, \([a, b]\), \( f'(t) \) exists a.e. Does it follow that

\[
f(x) - f(a) = \int_{a}^{x} f'(t) \, dt?
\]

Prove by giving an argument why this is so or else disprove by citing a counter example.

8. Show that in any Hilbert space there exists a maximal orthonormal set.

9. Let \( E \) be a Lebesgue measurable set in \( \mathbb{R} \). Suppose \( m(E) > 0 \). Consider the set

\[
E - E = \{ x - y : x \in E, y \in E \}.
\]

Show that \( E - E \) contains an interval. **Hint:** Let

\[
f(x) = \int \mathcal{N}_E(t) \mathcal{N}_E(x+t) \, dt.
\]

Show \( f \) is continuous at \( 0 \).

10. Suppose \( f \in L^2(\mathbb{R}^n) \) and that for all \( \phi \in C_c(\mathbb{R}^n) \), the space of continuous functions with compact support,

\[
\int_{\mathbb{R}^n} f(x) \phi(x) \, dm_n = 0
\]

Can it be concluded that \( f = 0 \) a.e.? Either prove or disprove.

11. Suppose \( \{ f_n \} \) is a sequence in \( L^2(\Omega) \) which converges weakly to \( f \in L^2(\Omega) \). This means

\[
(f, g)_{L^2(\Omega)} \rightarrow (f, g)_{L^2(\Omega)}
\]

for all \( g \in L^2(\Omega) \). Also suppose \( f_n \rightarrow f \) pointwise. Can it be concluded that \( f_n \rightarrow f \) in \( L^1(\Omega) \)? Here \((\Omega, \mathcal{F}, \mu)\) is a finite measure space, \( \mu(\Omega) < \infty \). Either prove or give a counter example.

12. Does there exist an uncountable set of irrational numbers which has no rational number as a limit point? Explain.

13. Give an example of a function which is continuous at every irrational number but discontinuous at every rational number. Also explain why there cannot be a function which is continuous at every rational number and discontinuous at every irrational number. For the second part, you might want to show the points of continuity of a function are a \( G_\delta \) set.

Complex Analysis

1. It is desired to find an analytic function, \( L(z) \) defined for all \( z \in \mathbb{C} \setminus \{0\} \) such that \( e^{L(z)} = z \). Is this possible? Explain why or why not.

2. If \( f \) is analytic, show that \( z \rightarrow \overline{f(z)} \) is also analytic.

3. Suppose that for some constants \( a, b \neq 0 \), \( a, b \in \mathbb{R} \), \( f(z + ib) = f(z) \) for all \( z \in \mathbb{C} \) and \( f(z + a) = f(z) \) for all \( z \in \mathbb{C} \). If \( f \) is analytic, show that \( f \) must be constant.

4. Suppose \( f \) is an entire function and that \( f \) has the property that whenever we write \( f(z) \) as a power series expanded about a point \( w \), it follows that at least one of the coefficients in the power series must equal zero. Show that \( f \) must be a polynomial. **Hint:** Define a set, \( A_n \) to be the points, \( w \) such that \( f(z) = \sum_{k=0}^{\infty} a_k (z - w)^k \), it follows \( a_n = 0 \). Thus \( A_n \) consists of the points where the power series of \( f \) centered at these points has the \( n^{th} \) coefficient equal to zero. Next show some \( A_n \) has a limit point.

5. We say a real valued function, \( u \) is subharmonic if \( u_{xx} + u_{yy} \geq 0 \). Show that if \( u \) is subharmonic on a bounded region, \( U \), and continuous on \( \overline{U} \) and \( u \leq m \) on \( \partial U \), then \( u \leq m \) on \( U \). **Hint:** If not, \( u \) achieves its maximum at \( (x_0, y_0) \in U \). Now consider \( u(x, y) = \varepsilon x^2 + u(x, y) \) where \( \varepsilon \) is small.

6. Consider the polynomial, \( z^{11} + 7z^5 + 3z^2 - 17 \). Does this polynomial have any zeros, \( z \) such that \( |z| > 2? \) **Hint:** Use Rouche’s theorem.

7. Evaluate \( \int_0^\infty \frac{\cos(ax)}{(x^2 + b^2)^2} \, dx \).

8. Prove Liouville’s theorem from the Cauchy integral formula.

9. Does there exist an entire function which maps \( \mathbb{C} \) onto the upper half plane? Explain.