1. A solution $x(t), 0 \leq t < \infty$, is called recurrent if $x(t_n) \to x(0)$ for some sequence $t_n \to \infty$. Prove that a gradient dynamical system has no nonconstant recurrent solution.

2. Sketch the phase portraits of
\[
\dot{x} = y, \\
\dot{y} = x - x^2 + \epsilon y
\]
when $\epsilon < 0, \epsilon = 0$ and $\epsilon > 0$.

3. Prove the following instability theorem: Let $V$ be a $C^1$ real-valued function defined on a neighborhood $U$ of an equilibrium point $\bar{x}$ of $x' = f(x)$. Suppose $V(\bar{x}) = 0$ and $V(x) > 0$ in $U - \bar{x}$. If there is a convergent sequence $x_n \to \bar{x}$ such that $V(x_n) > 0$, then $\bar{x}$ is unstable.

4. Let $U$ be an open subset of $\mathbb{R}^2$ and $V$ be an open subset of $U$ with $\bar{V} \subset U$. Let $f$ is a $C^1$ function on $U$. Prove that if $V$ contains a bounded positive orbit of $x' = f(x)$, then $U$ contains an equilibrium point.

5. Find the normal form (up to order 3) of the following equation
\[
\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x^2 + xy^2 \\ y^2 + x^3 \end{pmatrix}
\]

6. Consider the following boundary value problem
\[
x'' + \lambda x = f(\lambda, x), \quad 0 \leq t \leq \pi \\
x(0) = x(\pi) = 0.
\]
where $f: \mathbb{R}^2 \to \mathbb{R}$ is a $C^1$ function, $f(\lambda, 0) = 0$, $D_x f(\lambda, 0) = 0$, $\lambda \in \mathbb{R}$ is the bifurcation parameter. Find bifurcation points.

7. Consider $x' = f(x)$ and its perturbed equation $x' = f(x) + \epsilon h(t, x)$ where $x \in \mathbb{R}^n$, $f$ and $h$ are $C^2$ functions, and $h(t, x)$ is $T$-periodic in $t$, $0 < \epsilon$ is a parameter. Prove that if $x' = f(x)$ has a hyperbolic equilibrium point $p^*$, then the perturbed equation $x' = f(x) + \epsilon h(t, x)$ has a unique periodic solution $p(t, \epsilon)$ such that
\[
p(t, \epsilon) - p^* = O(\epsilon).
\]

8. Consider $x' = Ax + f(x)$ where $x \in \mathbb{R}^n$, $A$ is a $n \times n$ matrix, and $f$ is a $C^1$ function with $f(0) = 0$ and $f'(0) = 0$. Assume that the set of eigenvalues $\sigma(A) = \sigma_c \cup \sigma_s$, where $\sigma_c = \{ \lambda \in \sigma(A) \mid \Re \lambda = 0 \}$ and $\sigma_s = \{ \lambda \in \sigma(A) \mid \Re \lambda < 0 \}$. Let $\mathbb{R}^n = E_c \oplus E_s$ be the corresponding decomposition. We further assume that there exists a global center manifold
\[
W^c = \{ p + h(p) \mid p \in E_c \}
\]
where $h: E_c \to E_s$ is a $C^1$ function with $h(0) = 0$, $h'(0) = 0$ and $\text{Lip}(h) < 1$. Prove if $\dim(E_c) = 2$, then for each bounded solution $x(t, x_0)$ the omega limit set $\omega(x_0)$ of $x_0$ is a periodic orbit if the $\omega(x_0)$ contains no equilibrium point.