Instructions: Answer six questions, three from part A and three from part B. Use separate sheets of paper for each problem. Indicate clearly which problem you are doing. Hand in no more than the requested number of problems. State clearly any assumptions that you make including hypothesis and conclusion. Do not assume so much that the problem becomes trivial.

Policy on Misprints: The exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases do not interpret the problem in such a way that it becomes trivial.

Part A

1. A Hausdorff space $X$ is an $n$-manifold if every point $x \in X$ has a neighborhood homeomorphic to $\mathbb{R}^n$. Show that every compact connected 1-manifold is homeomorphic to the circle.

2. Prove that the countable product of compact spaces is compact.

3. Let $X$ be a Hausdorff space and suppose that $\{f_\alpha\}_{\alpha \in J}$ is a collection of continuous functions $f_\alpha : X \to [0, 1]$ satisfying the requirement that for each $x_0 \in X$ and each closed set $A$ with $x_0 \not\in A$, there is an index $\alpha$ with $f_\alpha(x_0) = 0$ and $f_\alpha(A) = \{1\}$. Then the function $F : X \to [0, 1]^J$ defined by $F(x) = (f_\alpha(x))_{\alpha \in J}$ is an embedding.

4. Use the Baire Category Theorem to show that if $C$ is a compact subset of the unit square $[0, 1] \times [0, 1]$ and if each of the vertical segments $\{x\} \times [0, 1]$ contains some nondegenerate segment $\{x\} \times [a_x, b_x]$ $(a_x < b_x)$ from $C$, then $C$ contains an open subset of $[0, 1] \times [0, 1]$.
Part B

1. Let $X$ be the wedge of two circles labeled by $a$ and $b$. Thus $\pi_1(X)$ is the free group $F$ on $a, b$. Define a homomorphism $h : F \to S_4$ by $h(a) = (2\ 3\ 4)$ and $h(b) = (1\ 4\ 3)$. Let $N < F$ be the kernel of $h$. Construct the covering space of $X$ corresponding to $N$.

2. Let $X$ be a connected, locally path connected space with $X = U_1 \cup U_2 \cup U_3$ where each $U_i$ is open and contractable and where $U_i \cap U_j$ are non-empty and contractable and pairwise disjoint. Calculate $\pi_1(X)$.

3. Let $p : X \to Y$ be a covering map, and $Z$ be a simply connected, path connected and locally path connected topological space. If $f : Z \to Y$ is a continuous function, show that there is a continuous function $\hat{f} : Z \to X$ with $p \circ \hat{f} = f$.

4. Let $A$ be the annulus in the plane $\mathbb{R}^2$ consisting of the set $\{(x, y)|1 \leq x^2 + y^2 \leq 4\}$. Let $S$ denote the surface obtained from the annulus by identifying antipodal points of the inner circle and by identifying antipodal points of the outer circle. Compute $\pi_1(S)$ and write $S$ as a connected sum of tori and projective planes.